University of Amsterdam

AMLAB Amsterdam Machine Learning Lab

Group Equivariant Deep Learning

Lecture 2 - Steerable group convolutions

Lecture 2.1 - Steerable kernels/basis functions

Definition and SO(2) example

Erik Bekkers, Amsterdam Machine Learning Lab, University of Amsterdam

This mini-course serves as a module with the UvA Master AI course Deep Learning 2 https://uvadl2c.github.io/

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Lecture 2 - Steerable group convolutions

[1]Worrall, D. E., Garbin, S. J., Turmukhambetov, D., & Brostow, G. J. Harmonic networks: Deep translation and rotation equivariance. CVPR 2017

Steerable basis

A vector $Y(x) = \begin{pmatrix} \vdots \\ Y_l(x) \\ \vdots \end{pmatrix} \in \mathbb{K}^L$ with (basis) functions $Y_l \in \mathbb{L}_2(X)$ is steerable if

$$\forall_{g \in G}: \qquad Y(g\,x) = \rho(g) Y(x),$$

where $g\,x$ denotes the action of G on X and $\rho(g) \in \mathbb{K}^{L \times L}$ is a representation of G.

I.e., we can transform all basis functions simply by taking a linear combination of the original basis functions.

Example: Steerable basis on S^1 (circular harmonics)

Basis functions (for $\mathbb{L}_2(S^1)$): $Y_l(\alpha) = e^{i\,l\,\alpha}$

Are steered by representations: $\rho_l(\theta) = e^{i\,l\,\theta}$

Proof: $Y_l(\alpha - \theta) = e^{i\,l\,(\alpha-\theta)}$
$= e^{-i\,l\,\theta}\, e^{i\,l\,\alpha}$
$= \rho_l(-\theta)\, Y_l(\alpha)$

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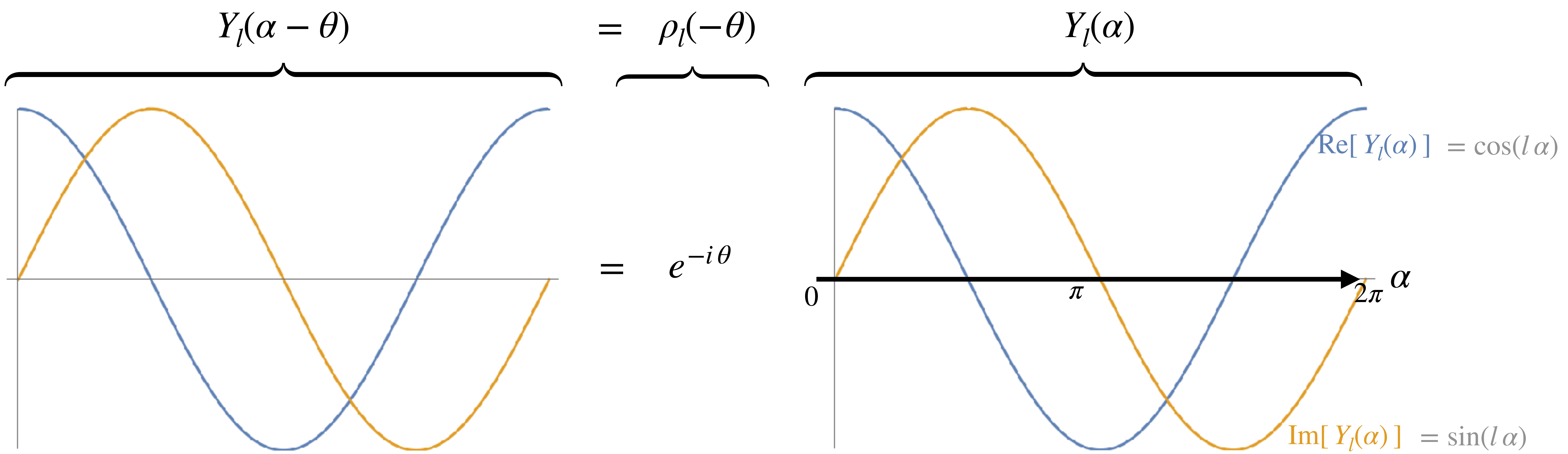


Example: Steerable basis on S^1 (circular harmonics)

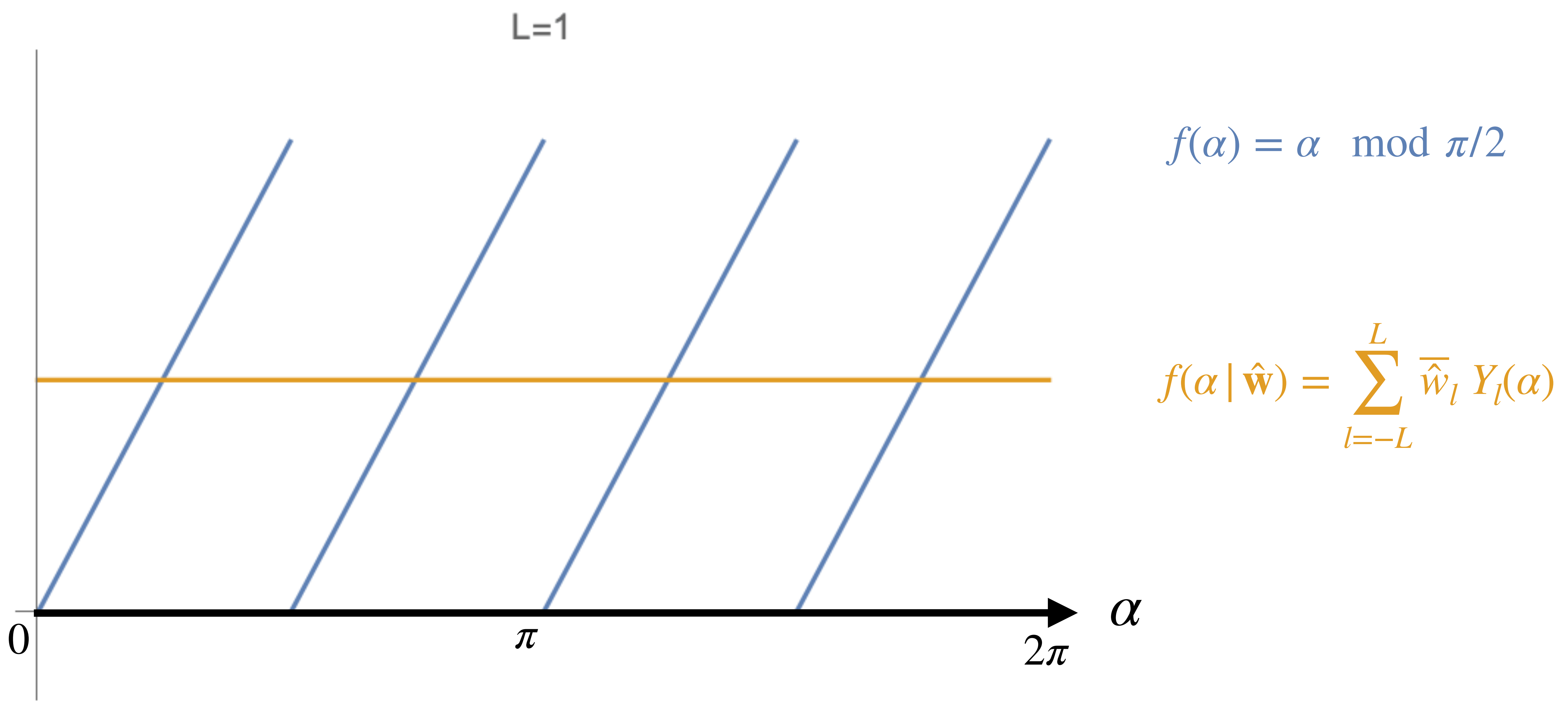
Basis functions (for $\mathbb{L}_2(S^1)$):

$$Y_l(\alpha) = e^{i\,l\,\alpha}$$

Form a complete orthonormal (Fourier) basis:

Y_l *are given by the irreps of* $SO(2)$ *and hence form orthogonal basis (Peter-Weyl Theorem)*

$$f(\alpha \,|\, \hat{\mathbf{w}}) = \sum_{l=-\infty}^{\infty} \overline{\hat{w}_l}\, Y_l(\alpha)$$



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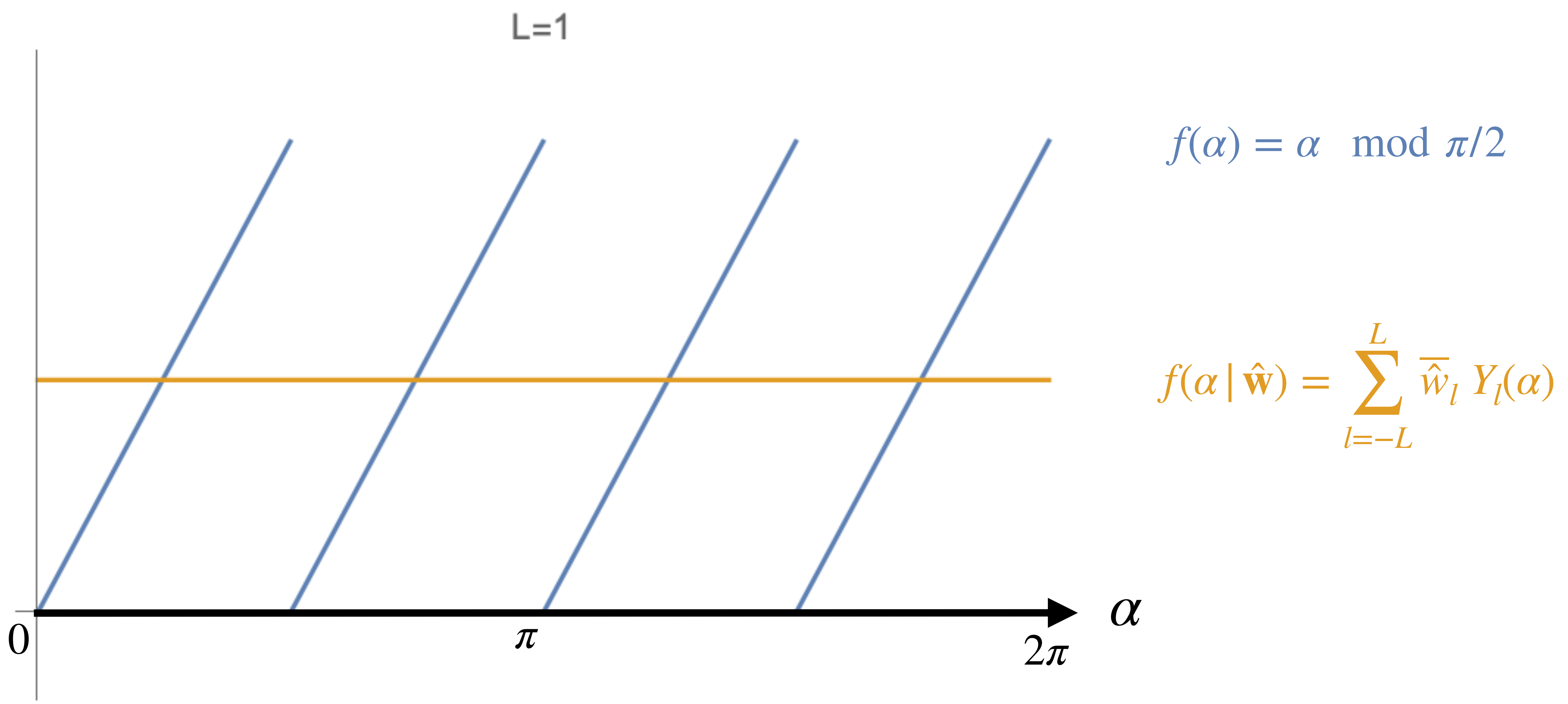

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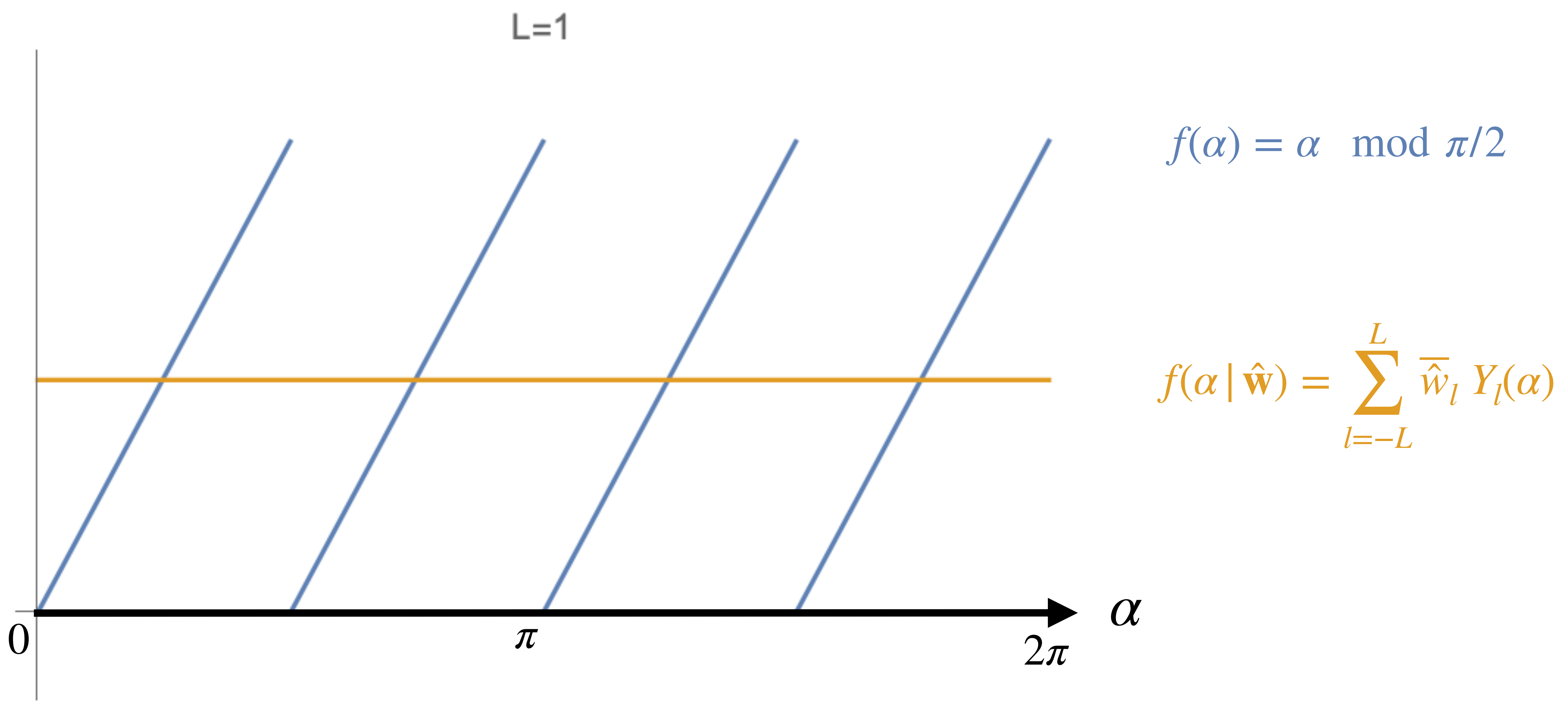

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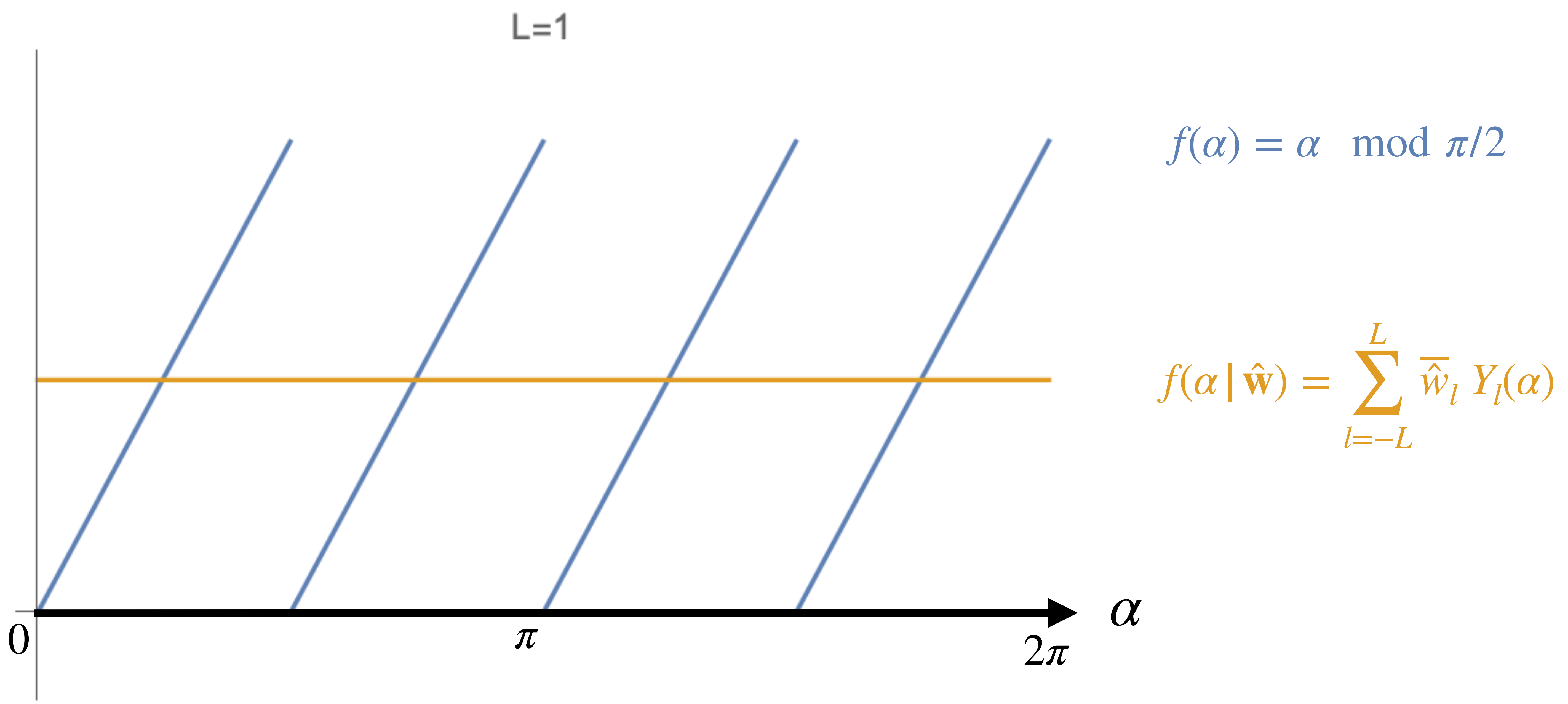

Example: Steerable basis on S^1 (circular harmonics)

Basis functions (for $\mathbb{L}_2(S^1)$): $$Y_l(\alpha) = e^{i l \alpha}$$

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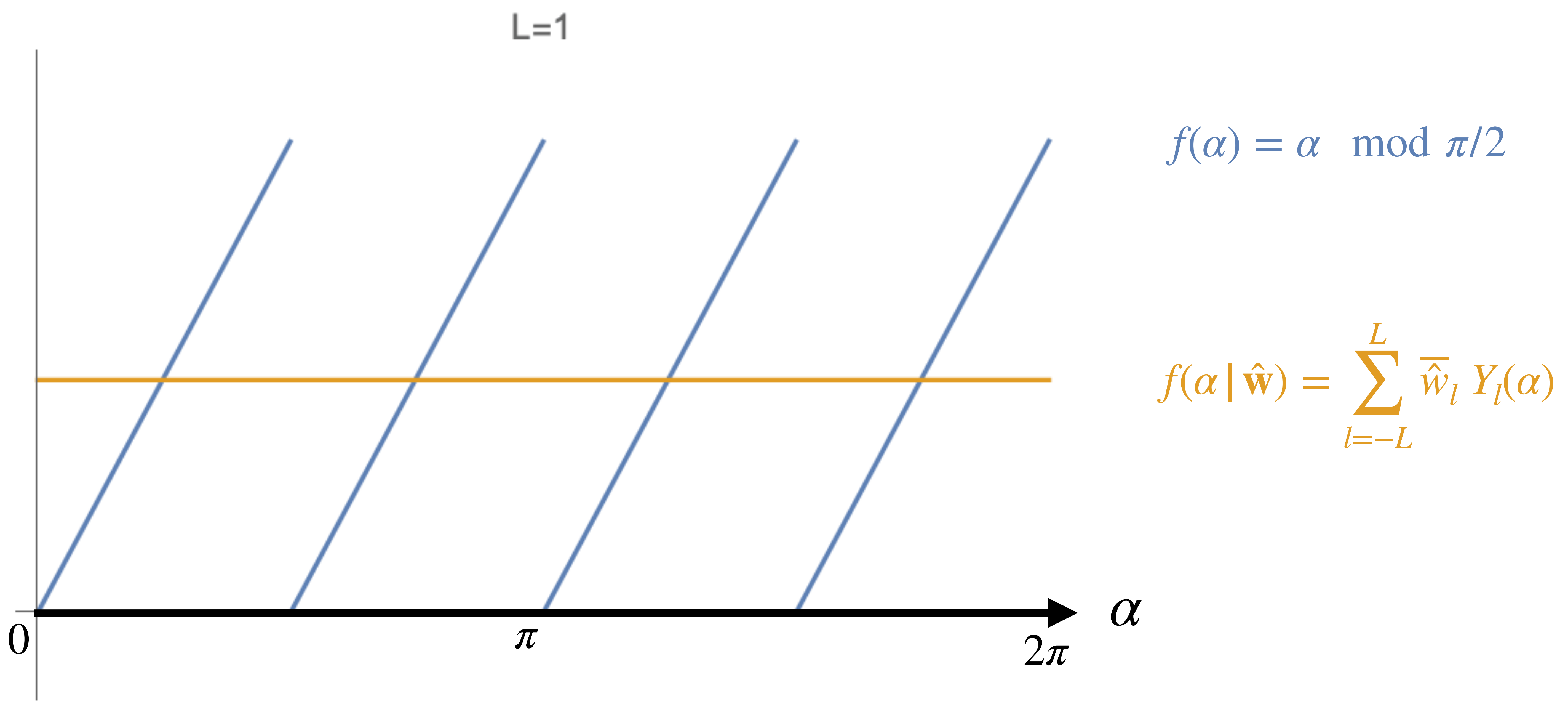
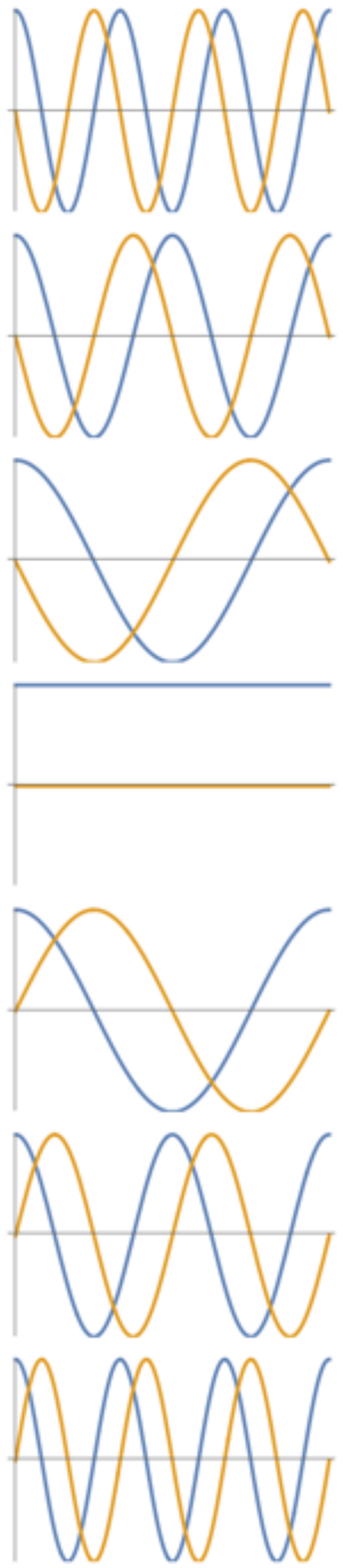


Example: Steerable basis on S^1 (circular harmonics)

$Y(\alpha - \theta)$

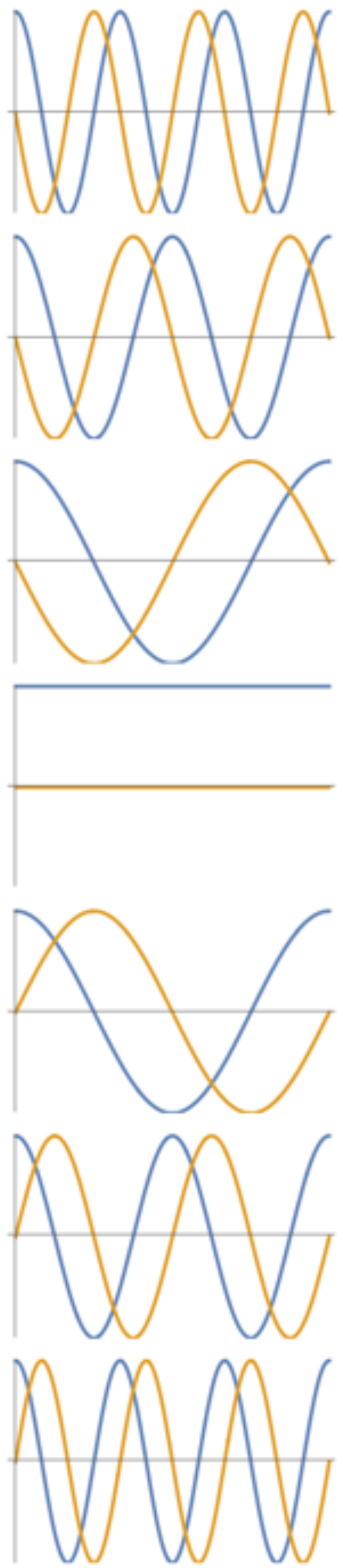


$$=$$

$$\rho(-\theta) = \bigoplus_{l=-L}^{L} \rho_l(-\theta)$$

$$\begin{pmatrix} e^{i3\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i2\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i1\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i1\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-i2\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-i3\theta} \end{pmatrix}$$

$Y(\alpha)$



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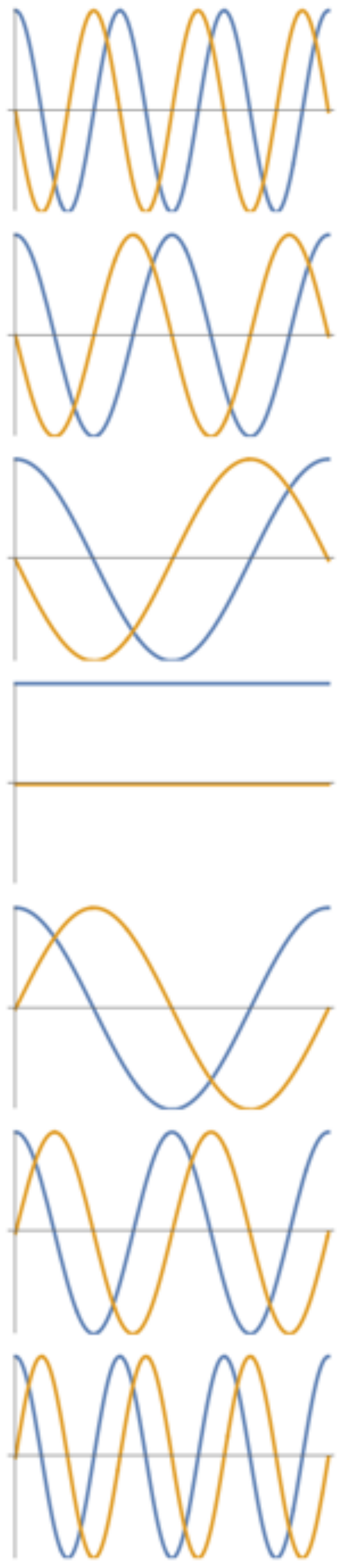

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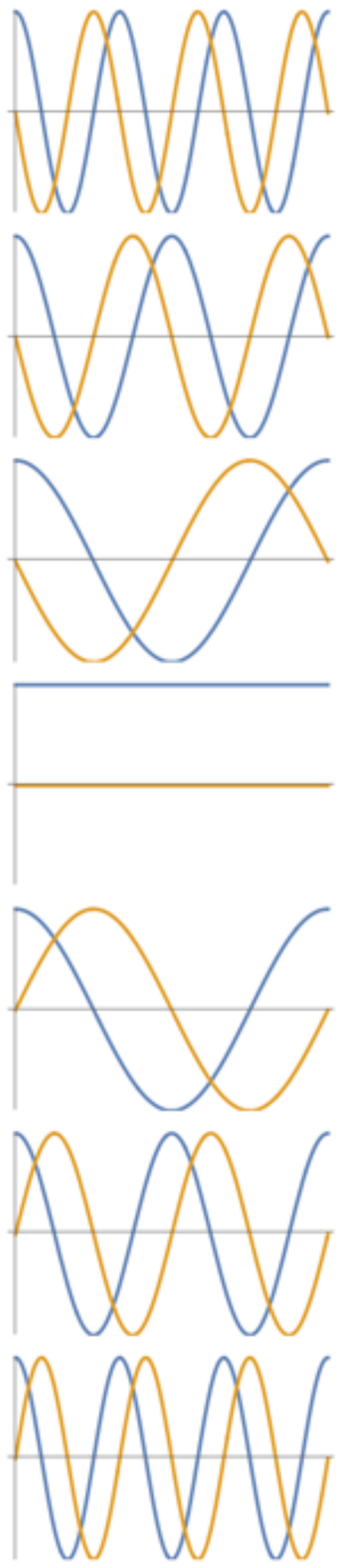


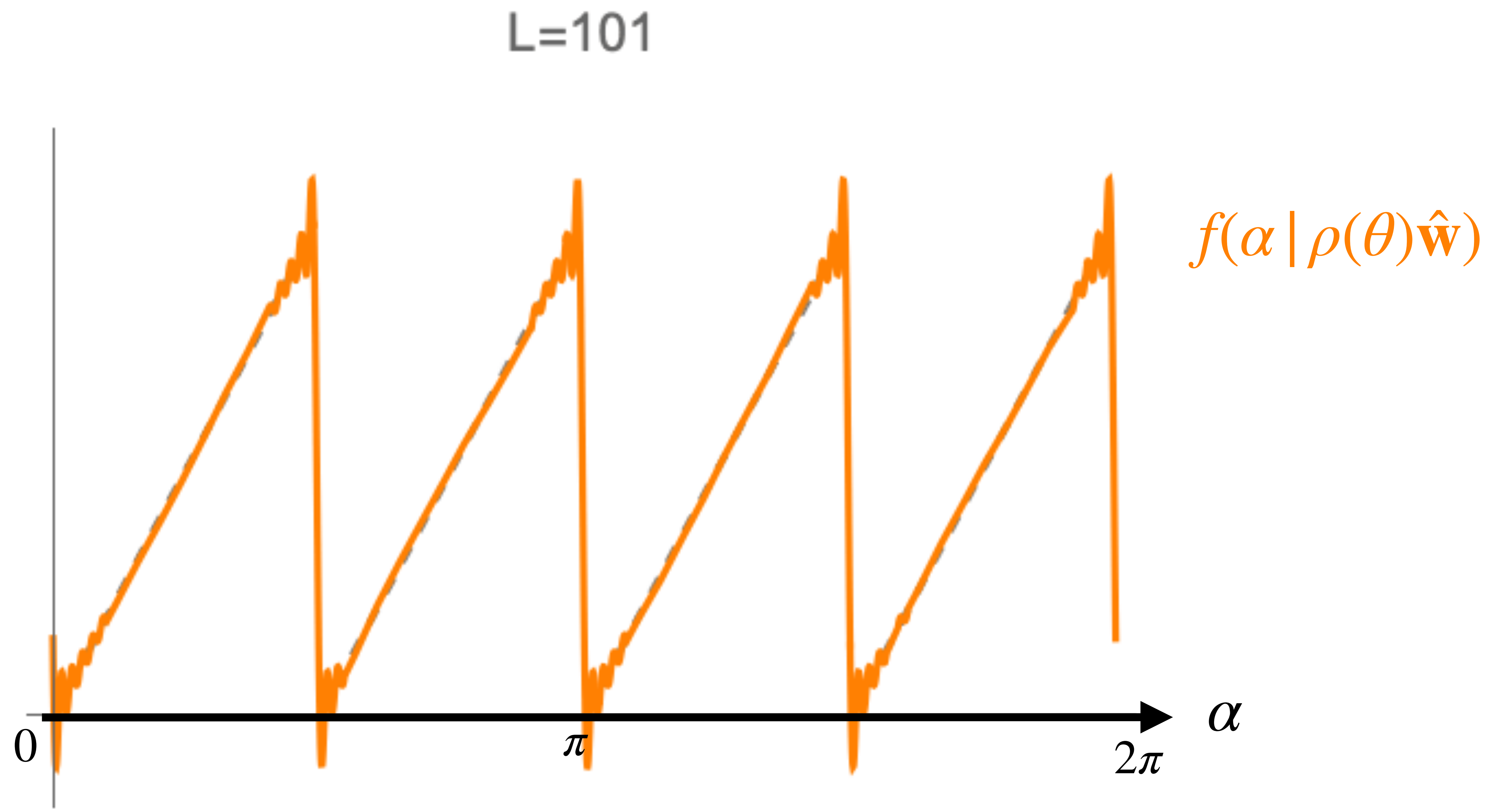
Example: Steerable basis on S^1 (circular harmonics)

Let $$f(\alpha \mid \hat{\mathbf{w}}) = \hat{\mathbf{w}}^\dagger Y(\alpha)$$

Then we can steer/shift this function by transforming the weights $\hat{\mathbf{w}}$

$$f(\alpha - \theta \mid \hat{\mathbf{w}}) = f(\alpha \mid \rho(\theta)\hat{\mathbf{w}})$$

Proof:
$$\begin{aligned} f(\alpha - \theta \mid \hat{\mathbf{w}}) &= \hat{\mathbf{w}}^\dagger Y(\alpha - \theta) \\ &= \hat{\mathbf{w}}^\dagger \rho(-\theta) Y(\alpha) \\ &= \hat{\mathbf{w}}^\dagger \rho(\theta)^\dagger Y(\alpha) \\ &= (\rho(\theta)\hat{\mathbf{w}})^\dagger Y(\alpha) \\ &= f(\alpha \mid \rho(\theta)\hat{\mathbf{w}}) \end{aligned}$$



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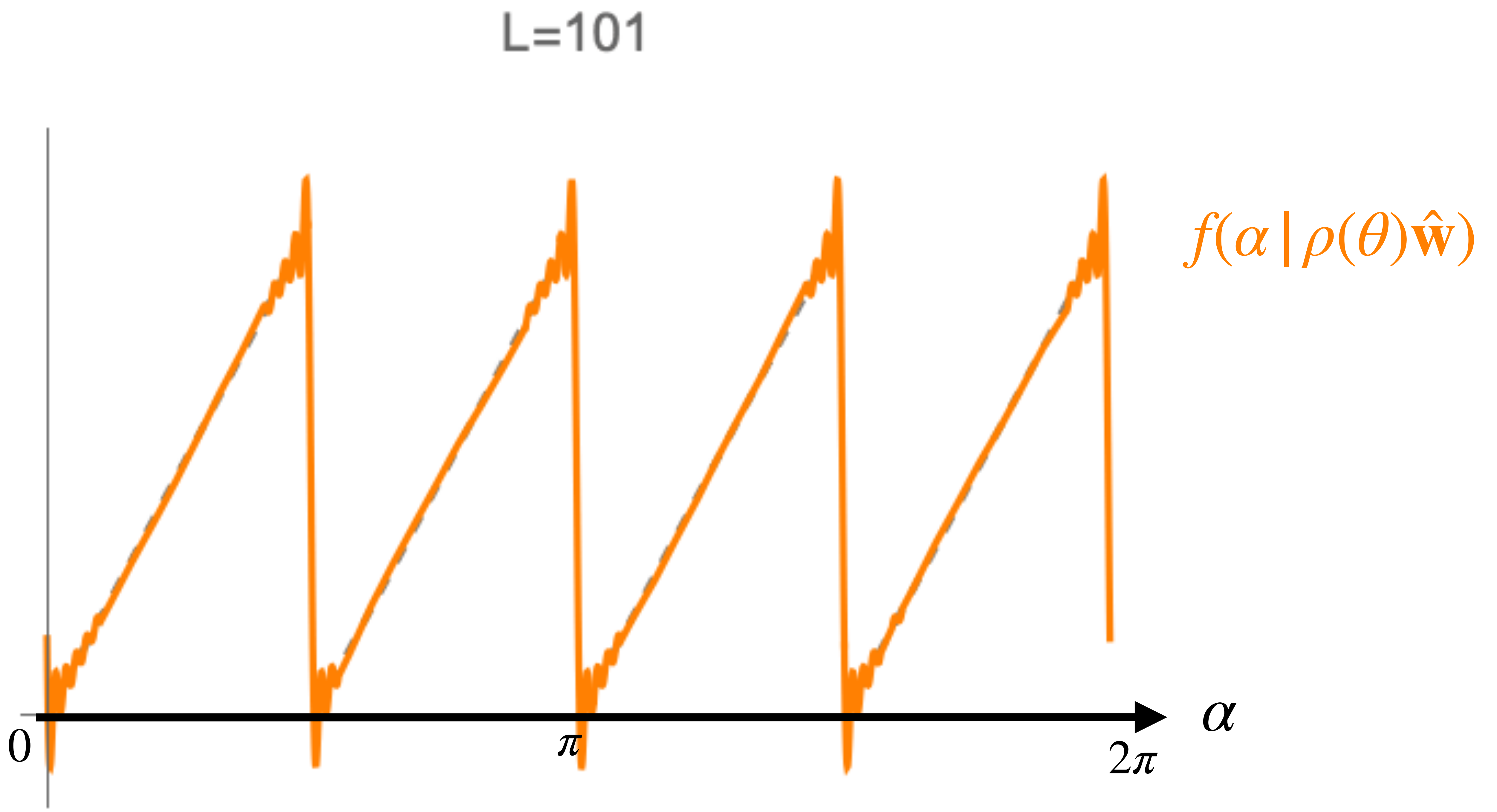
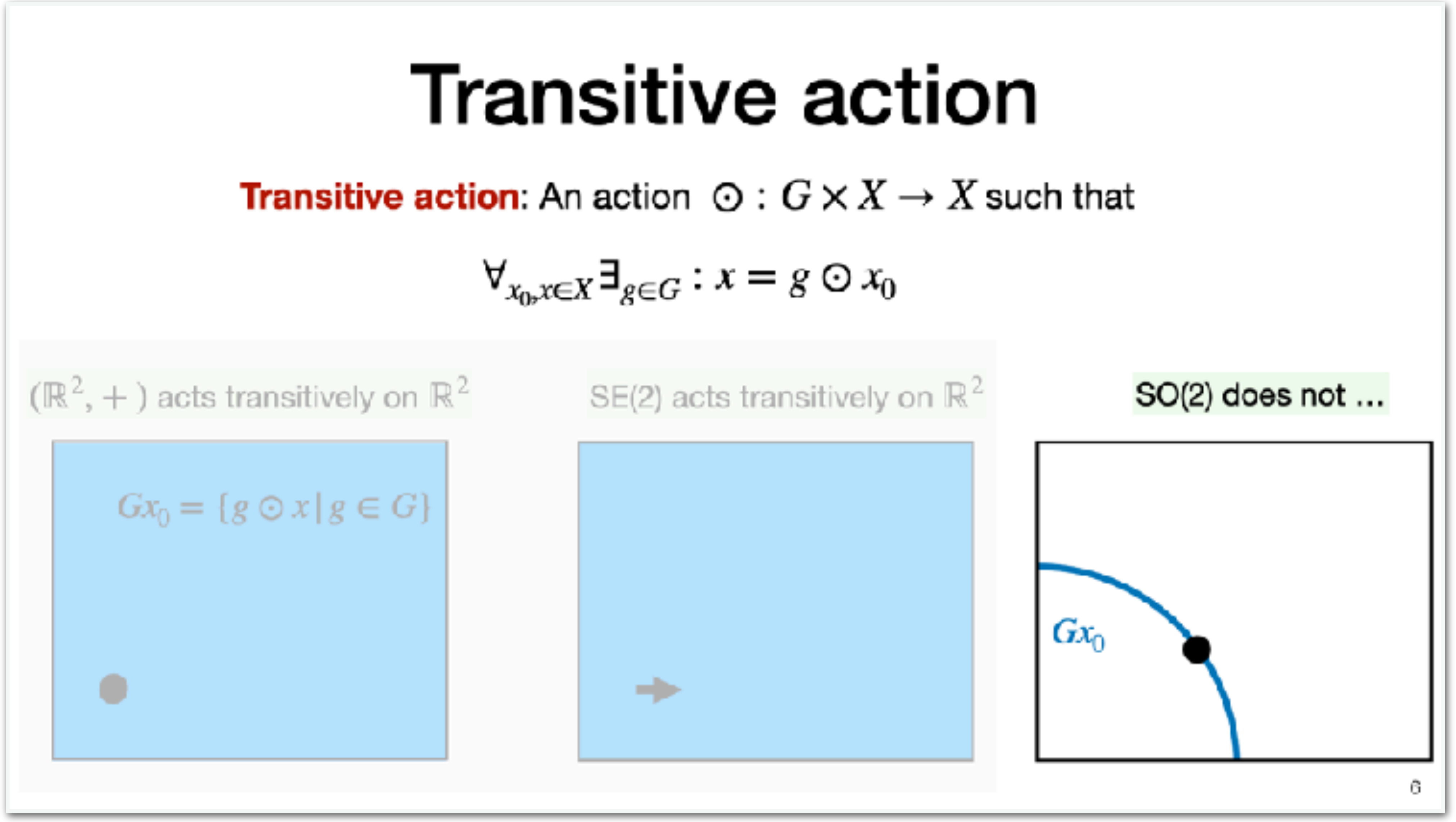


Two dimensional rotation-steerable functions

- The previous functions $\rho_l(\theta) = e^{il\theta}$ are (irreducible) representations of $SO(2)$

Recall lecture 1.6 (Group Theory | Homogeneous/quotient spaces)



- The group $SO(2)$ can also act on $\mathbb{R}^2$
  - Though not transitively…
  - It does act transitively on S^1 though

- Use polar coordinates $\mathbb{R}^2 \ni \mathbf{x} \quad \leftrightarrow \quad (r, \alpha) \in \mathbb{R}^+ \times S^1$ to come up with a rotation-steerable basis for $\mathbb{L}_2(\mathbb{R}^2)$!

Two dimensional rotation-steerable functions

- Consider a function $f(\mathbf{x}) = \bar{f}(r, \alpha)$ in polar coordinates

$$\mathbf{x} = (r\cos\alpha, r\sin\alpha)$$

- The action of $SO(2)$ on $\mathbb{R}^2$ in polar coords translates to

$$\mathbf{x} \mapsto \mathbf{R}_\theta \mathbf{x} \quad \leftrightarrow \quad (r, \alpha) \mapsto (r, \alpha + \theta)$$

Proof: $\mathbf{R}_\theta \mathbf{x} = \mathbf{R}_\theta \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}$

$$= \begin{pmatrix} r(\cos\theta\cos\alpha - \sin\alpha\sin\theta) \\ r(\cos\theta\sin\alpha + \cos\alpha\sin\theta) \end{pmatrix}$$

$$= \begin{pmatrix} r\cos(\theta + \alpha) \\ r\sin(\theta + \alpha) \end{pmatrix}$$

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- Then, functions are rotated simply by a shift in the angular axis

$$\mathscr{L}_\theta^{SO(2)} f(\mathbf{x}) = f(\mathbf{R}_\theta^{-1}\mathbf{x}) \quad \leftrightarrow \quad \mathscr{L}_\theta^{SO(2)} \bar{f}(r, \alpha) = \bar{f}(r, \alpha - \theta)$$

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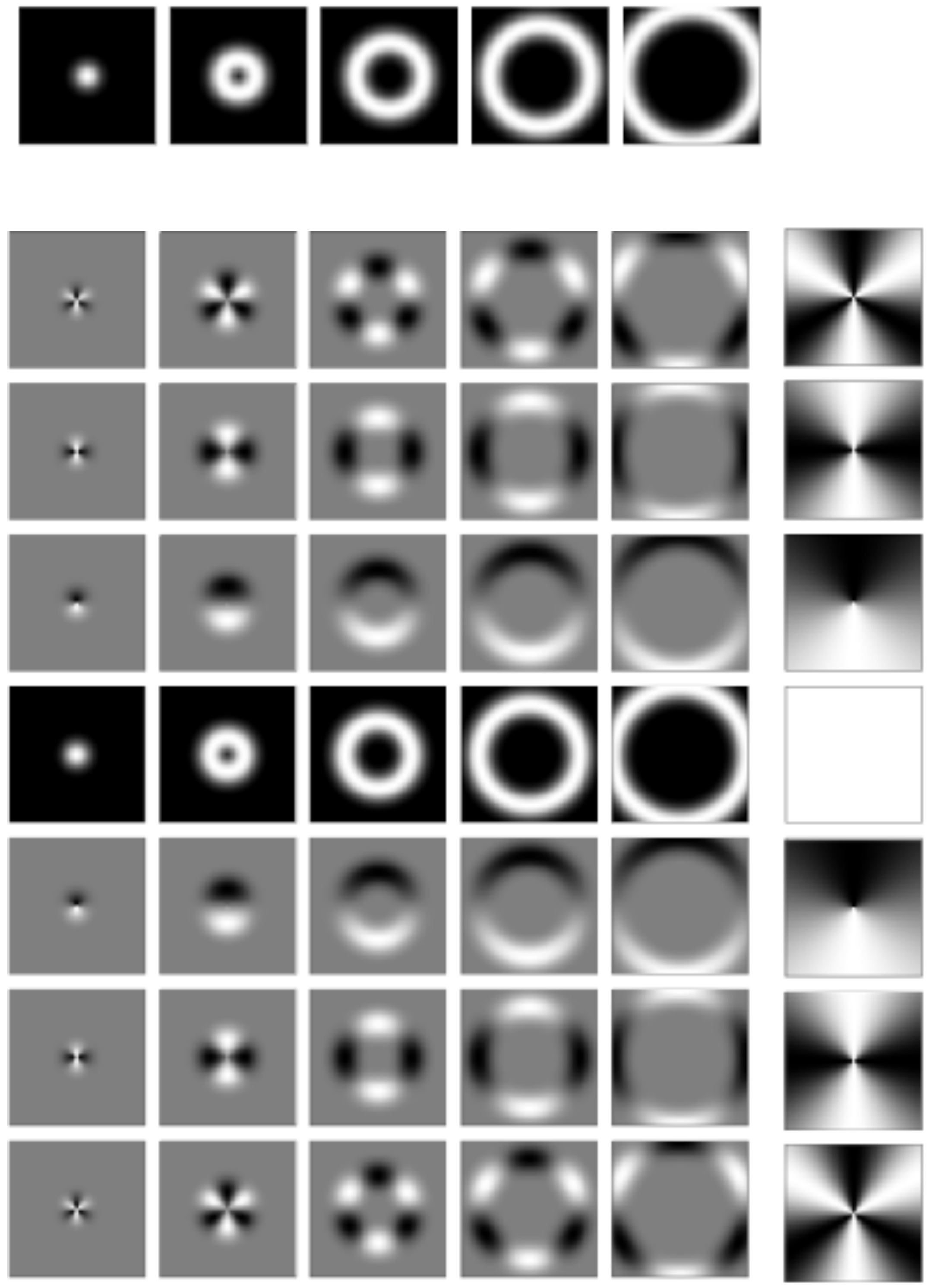
- Now, let's use this to paramatrize polar-separable conv kernels and focus on the angular part

$$k(\mathbf{x}\,|\,\mathbf{w}) = k^{\rightarrow}(r\,|\,\mathbf{w})\, k^{\circlearrowleft}(\alpha\,|\,\mathbf{w})$$

A function on S^1 !!!

$k_m^{\rightarrow}(r)$



$k_l^{\circlearrowleft}(\alpha)$

Two dimensional rotation-steerable functions

- Consider polar-separable convolution kernel:

$$k(\mathbf{x} \,|\, \mathbf{w}) = k^{\rightarrow}(r \,|\, \mathbf{w})\, k^{\circlearrowleft}(\alpha \,|\, \mathbf{w}),$$

- with $k^{\circlearrowleft}$ in an $SO(2)$ steerable basis, and $k^{\rightarrow}$ in some radial basis:

$$k^{\circlearrowleft}(\alpha \,|\, \mathbf{w}) = \sum_{l} \overline{w}_l Y_l(\alpha), \qquad \text{e.g., with} \qquad Y_l(\alpha) = e^{i\,l\,\alpha},$$

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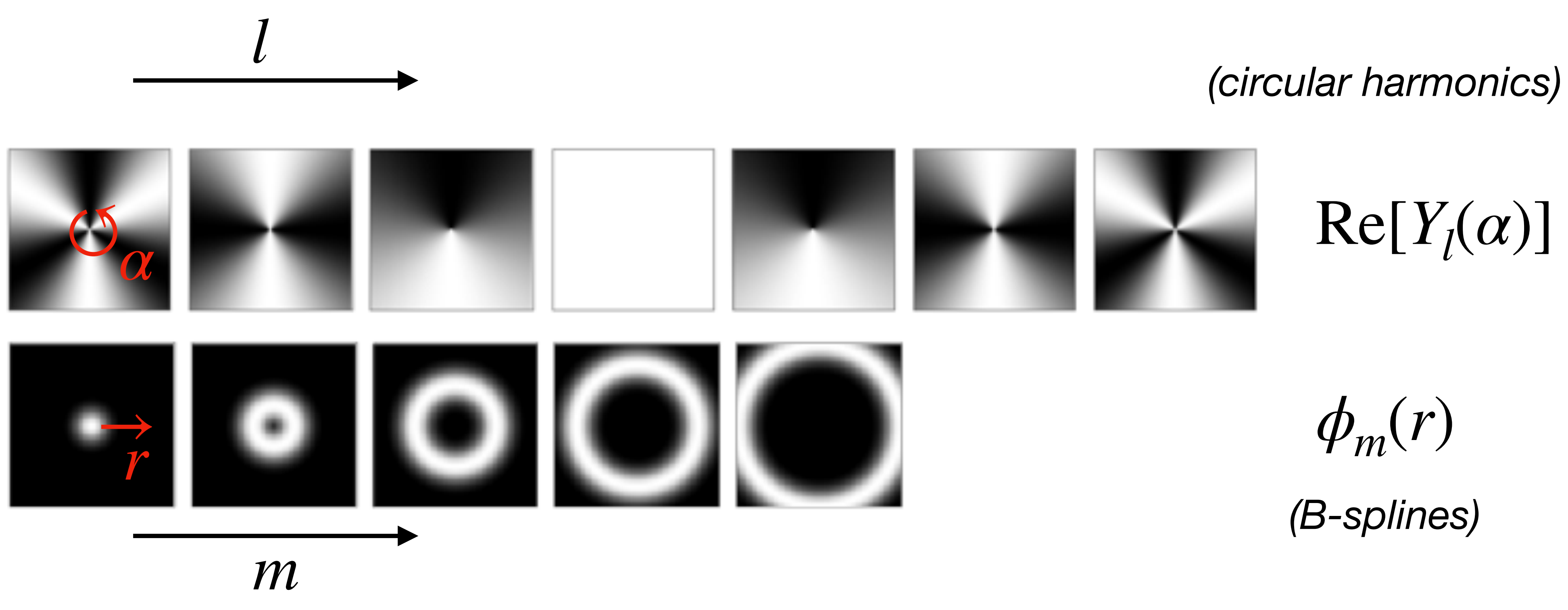

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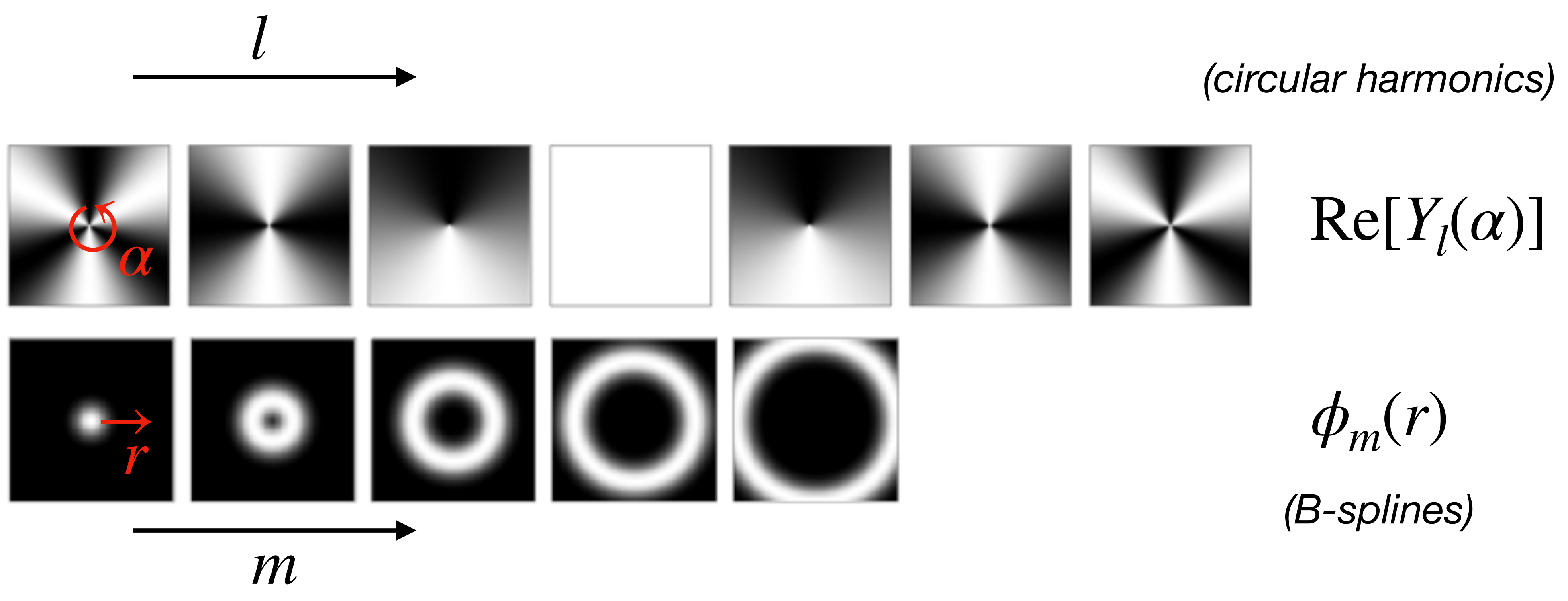

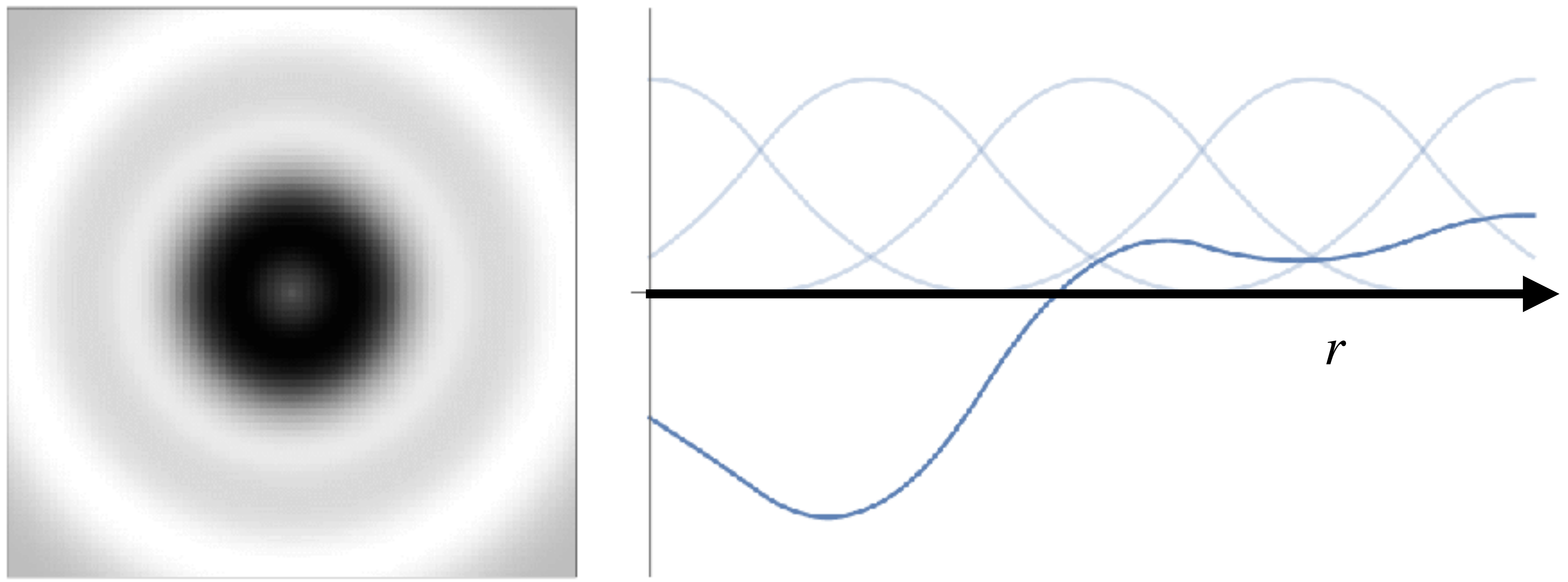

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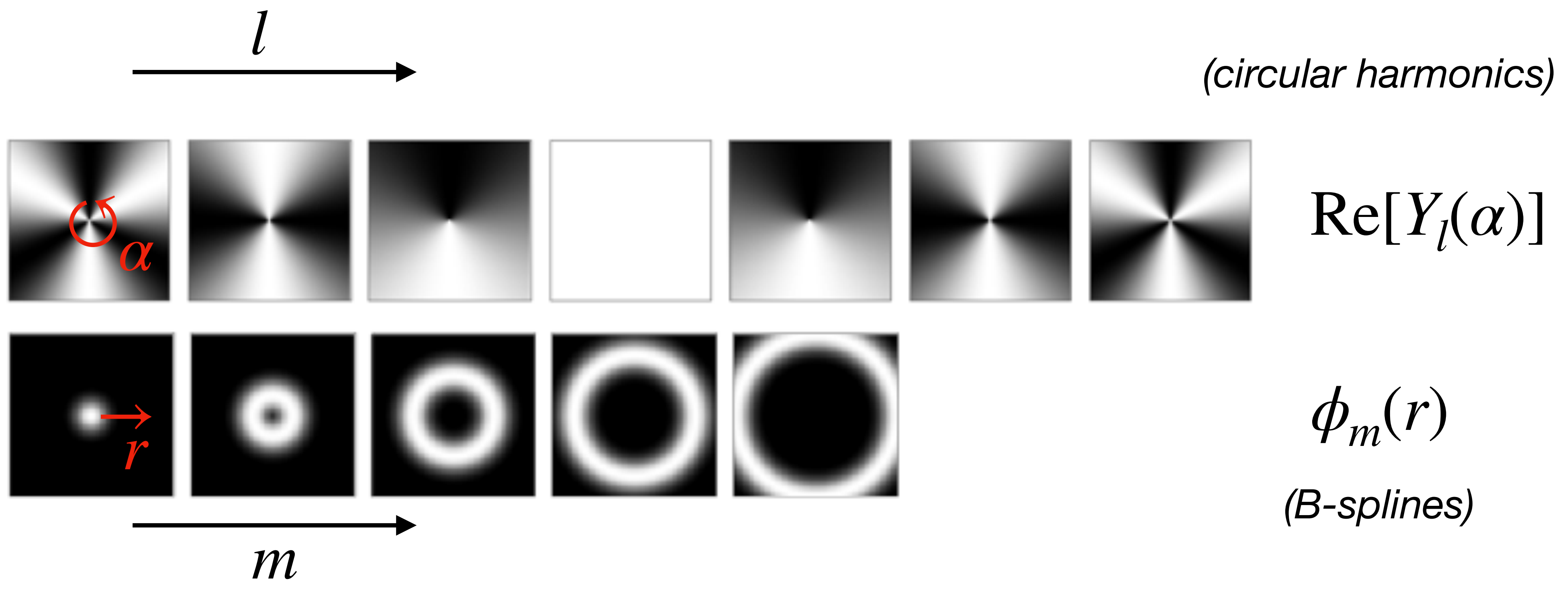

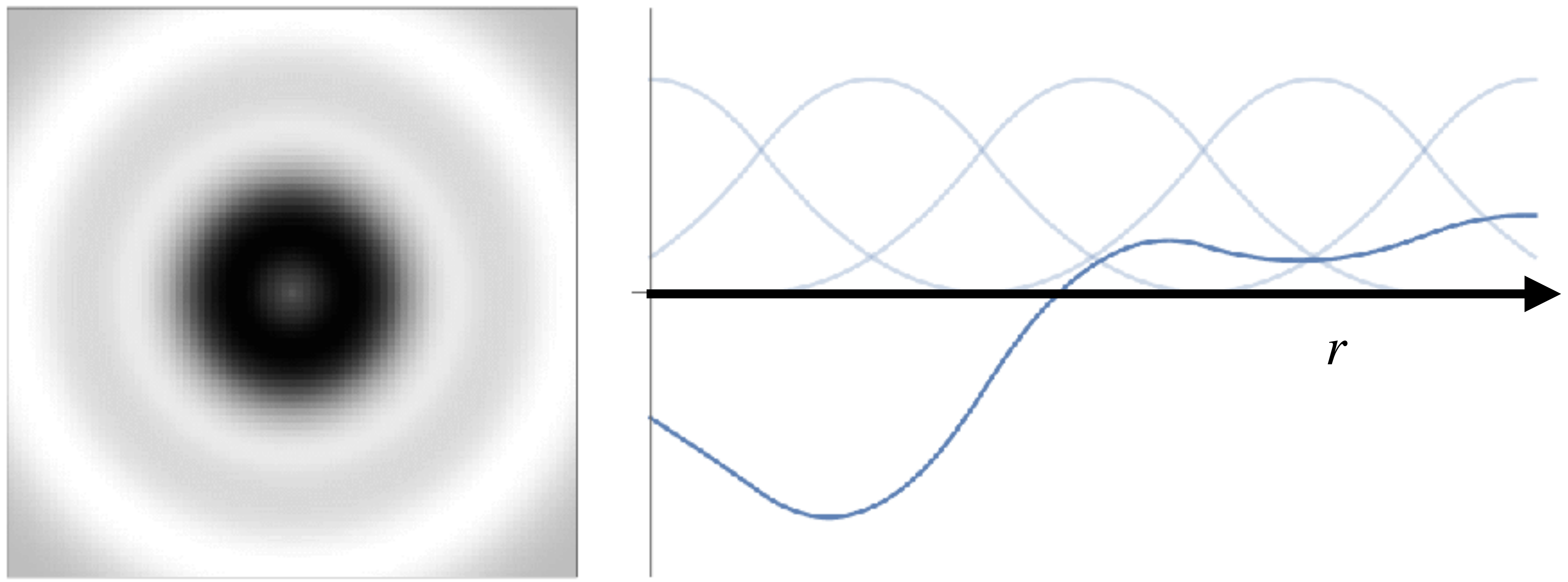

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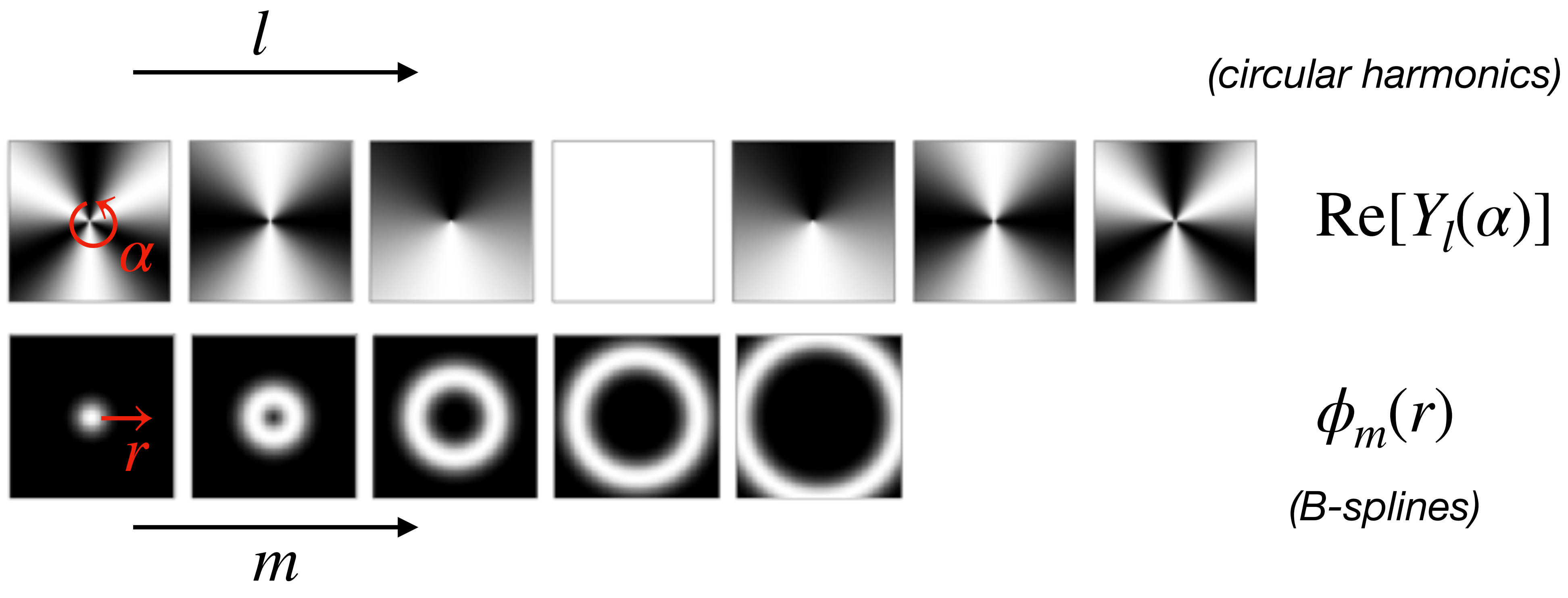

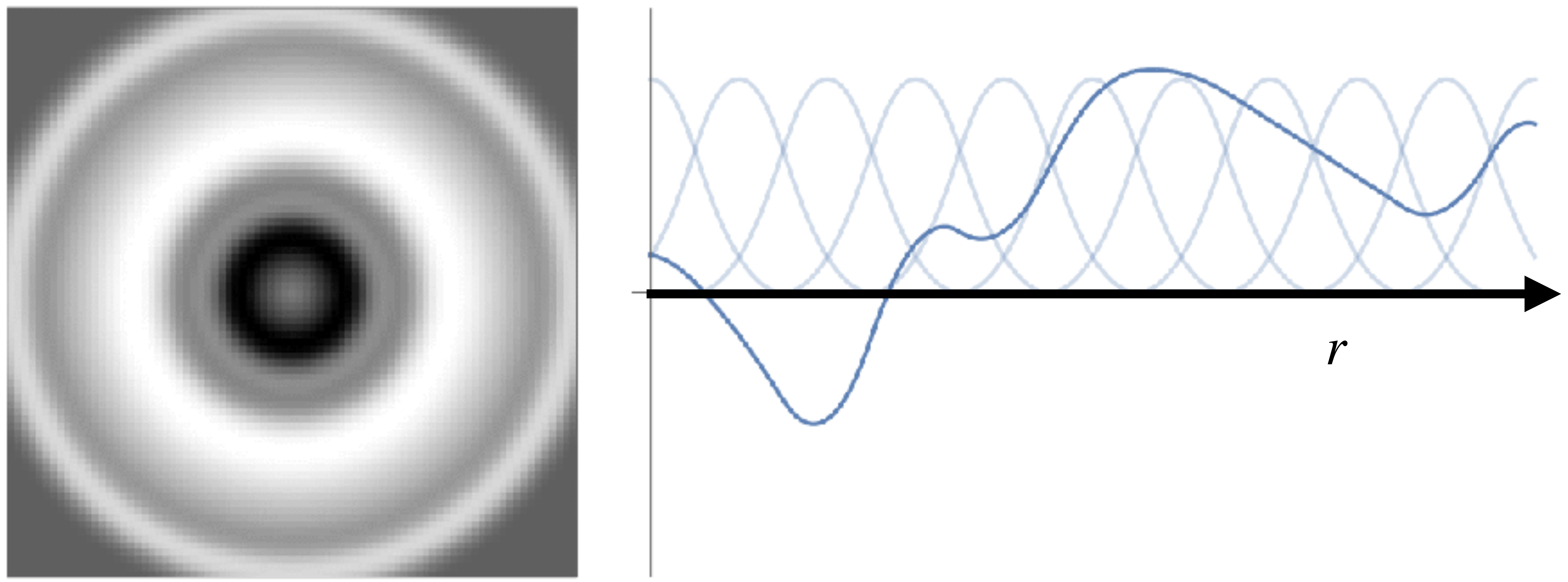

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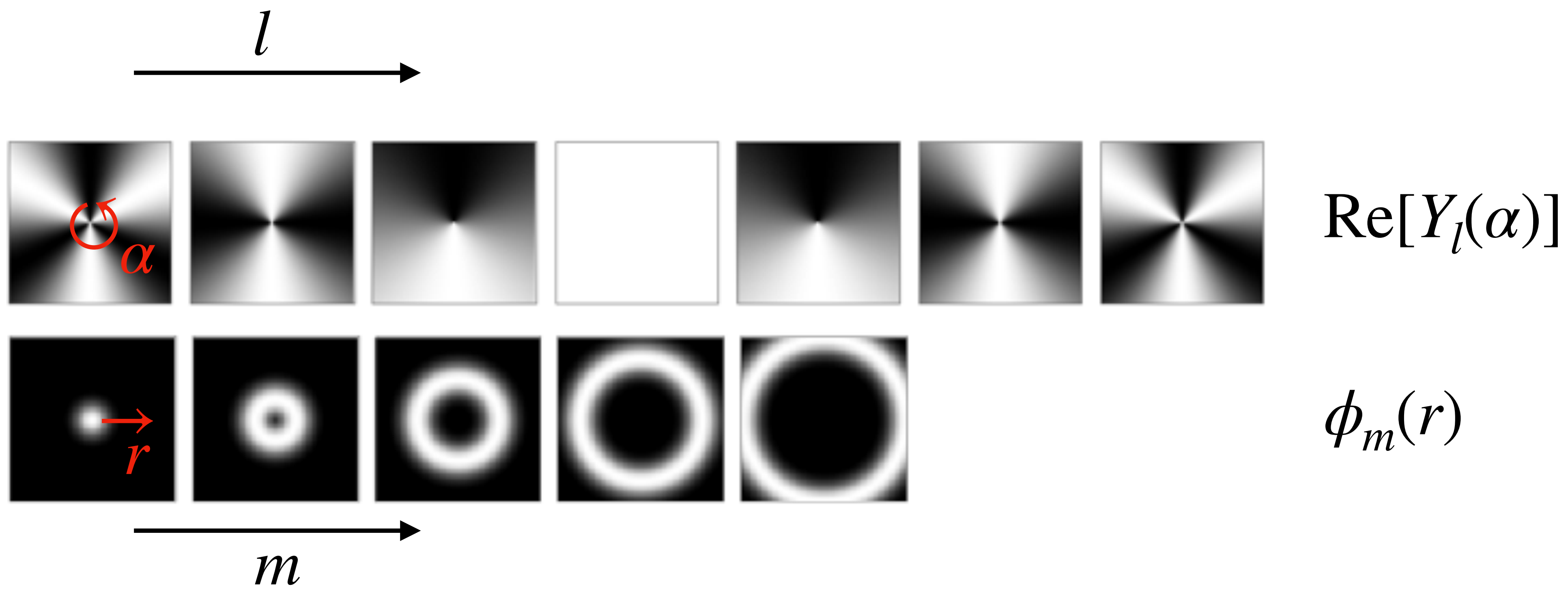

- Then we may as well write it as

$$k(\mathbf{x}\,|\,\mathbf{w}) = \sum_l \sum_m w_m \overline{w}_l\, \phi_m(r)\, Y_l(\alpha)$$

$$= \sum_l \sum_m \overline{w}_{ml}\, \phi_m(r)\, Y_l(\alpha) \qquad \text{("absorb" weights)}$$

$$= \sum_l \overline{\hat{w}}_l(r)\, Y_l(\alpha) \qquad \text{with radius dependent weights } \hat{w}_l(r) = \sum_m w_{ml}\, \phi_m(r)$$

- Then such kernel is clearly rotation steerable!

$$k(\mathbf{R}_\theta^{-1}\mathbf{x}\,|\,\hat{\mathbf{w}}(r)) = k(\mathbf{x}\,|\,\rho(\theta)\hat{\mathbf{w}}(r))$$

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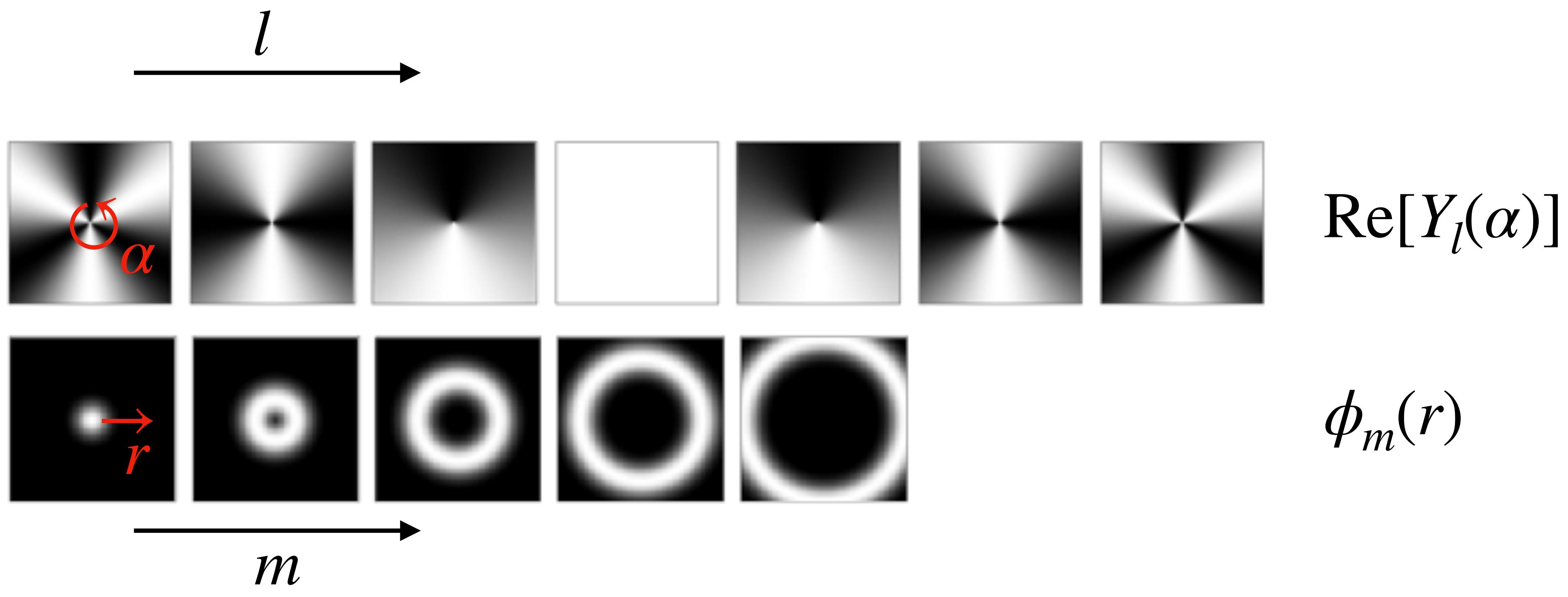

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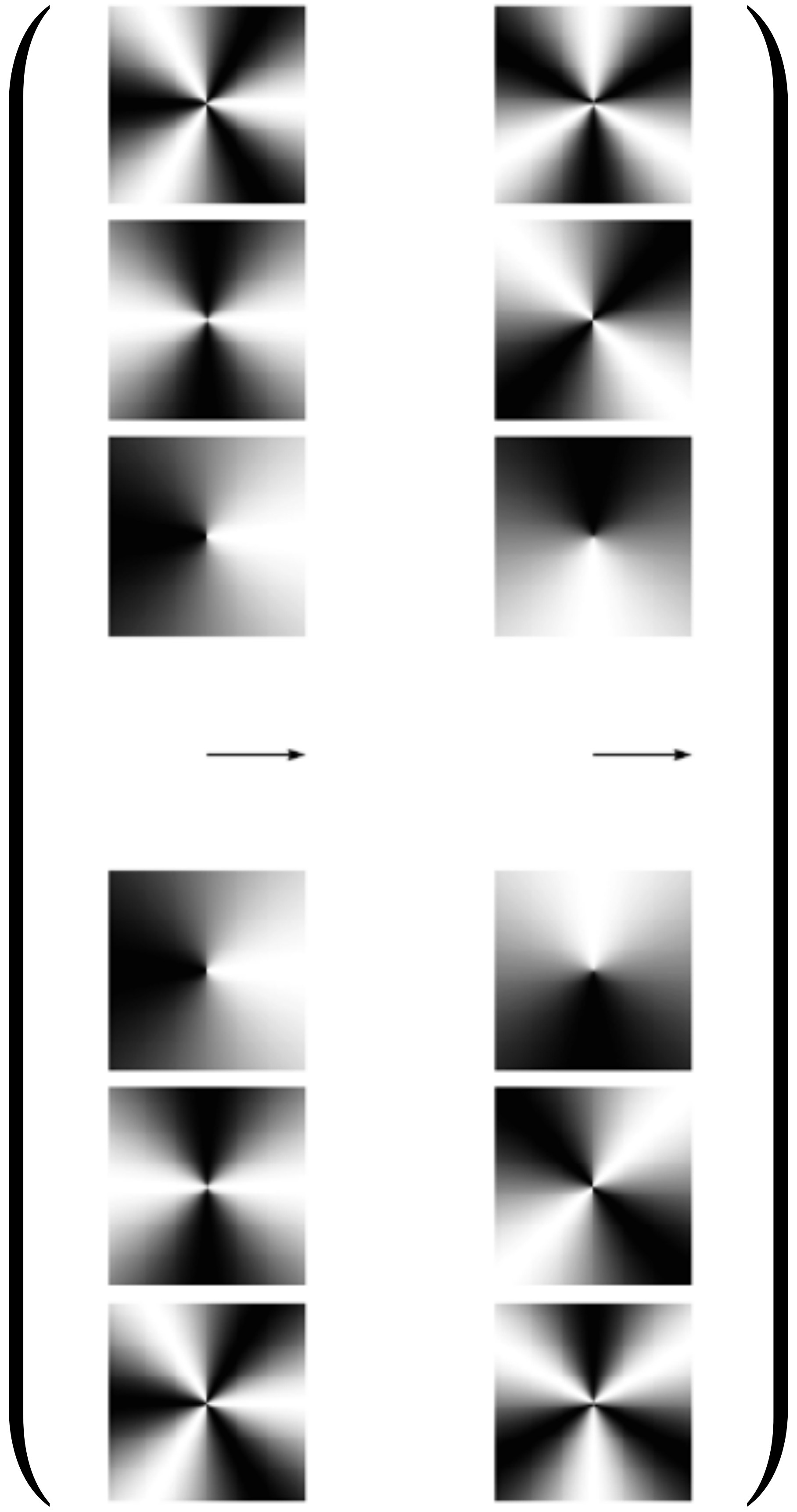
$$k(\mathbf{R}_\theta^{-1}\mathbf{x}\,|\,\hat{\mathbf{w}}(r)) = k(\mathbf{x}\,|\,\rho(\theta)\hat{\mathbf{w}}(r))$$

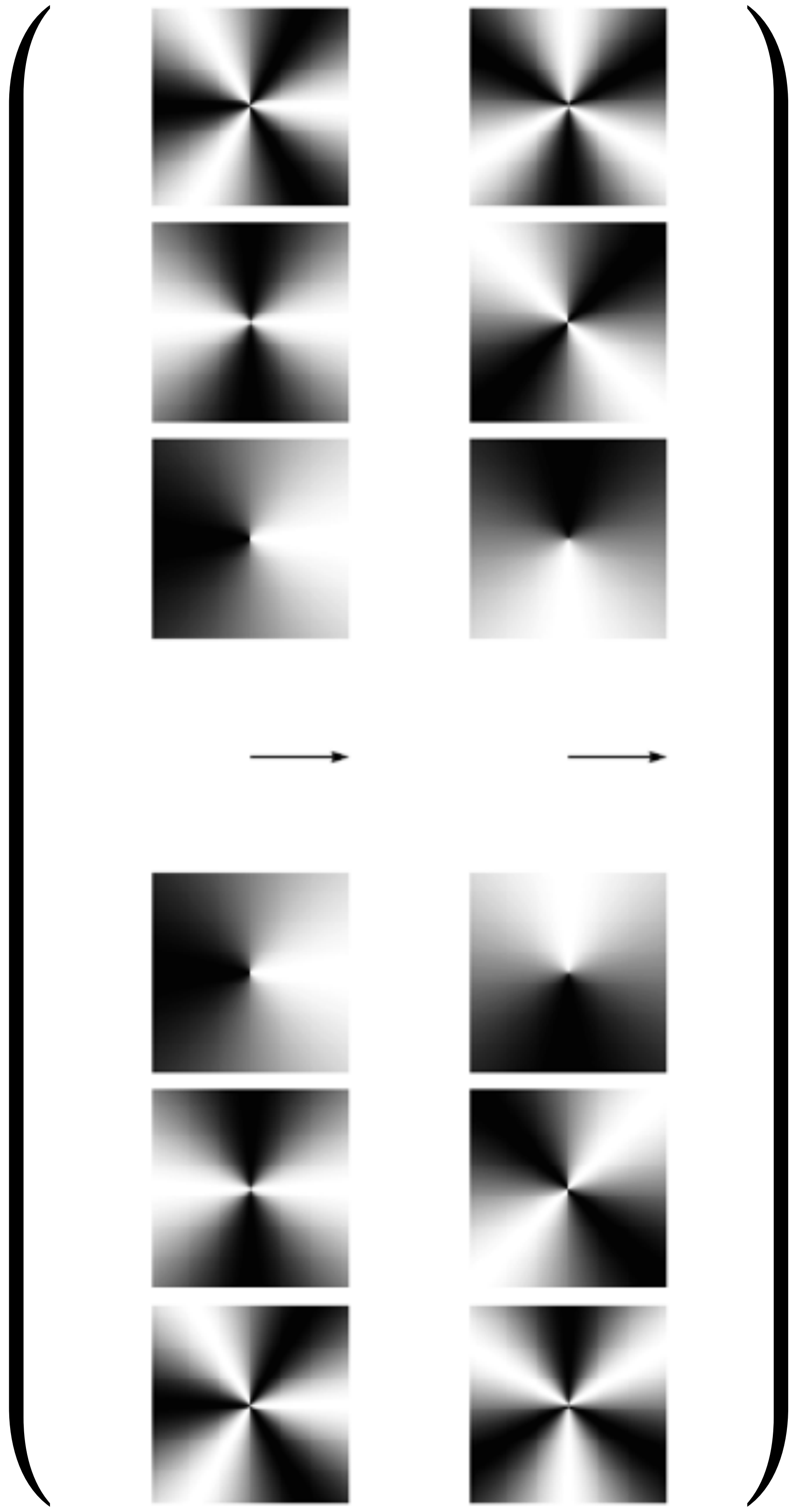
Or directly parametrize as $\hat{\mathbf{w}}(r) = \mathrm{MLP}(r\,|\,\mathbf{w})$!

Complex (irreducible) representations

$$Y(\mathbf{R}_\theta^{-1}\,\mathbf{x}) = \rho(\mathbf{R}_\theta^{-1})\quad Y(\mathbf{x})$$



$$= \begin{pmatrix} e^{3i\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{1i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-1i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-3i\theta} \end{pmatrix}$$



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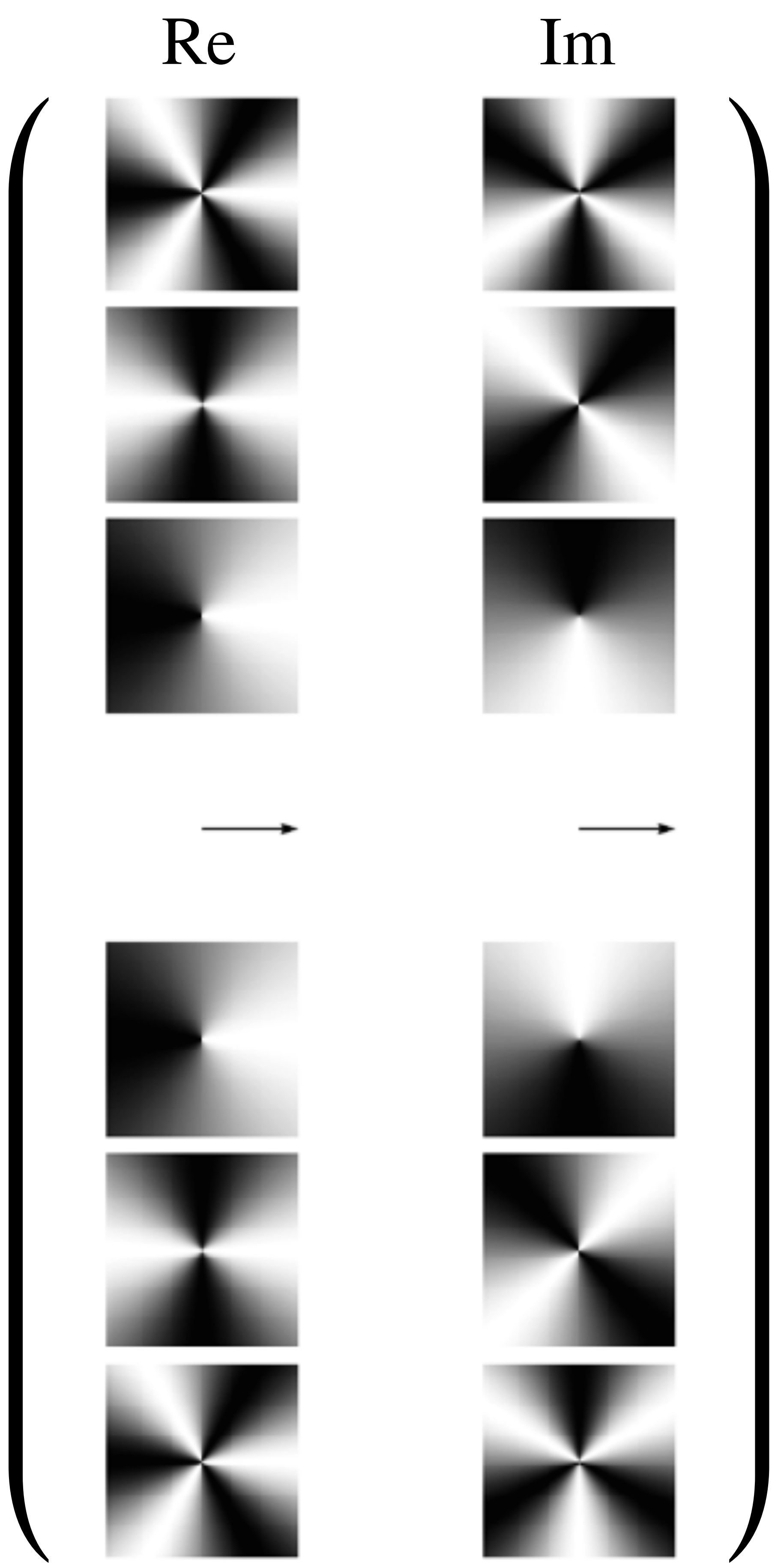

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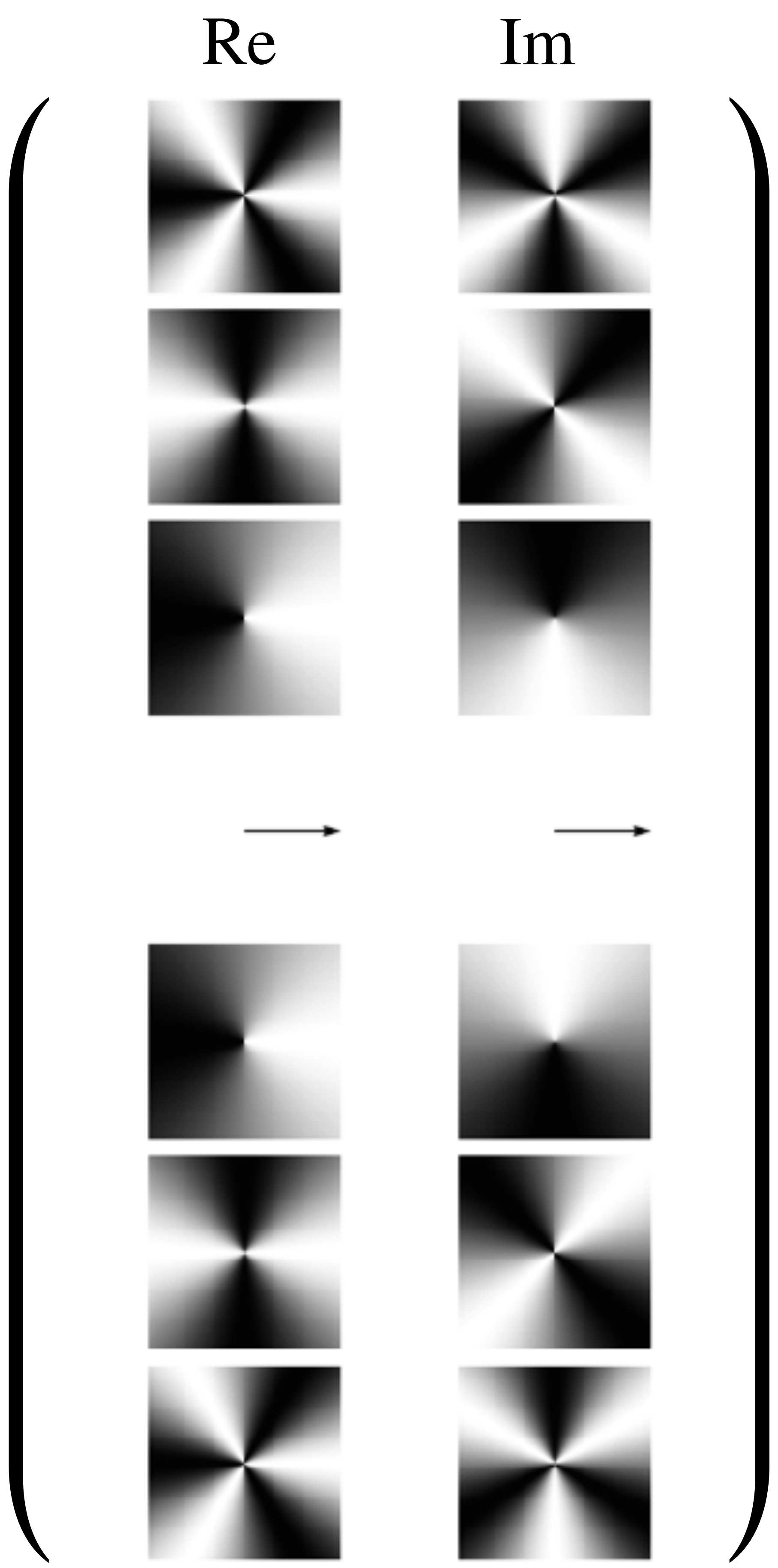
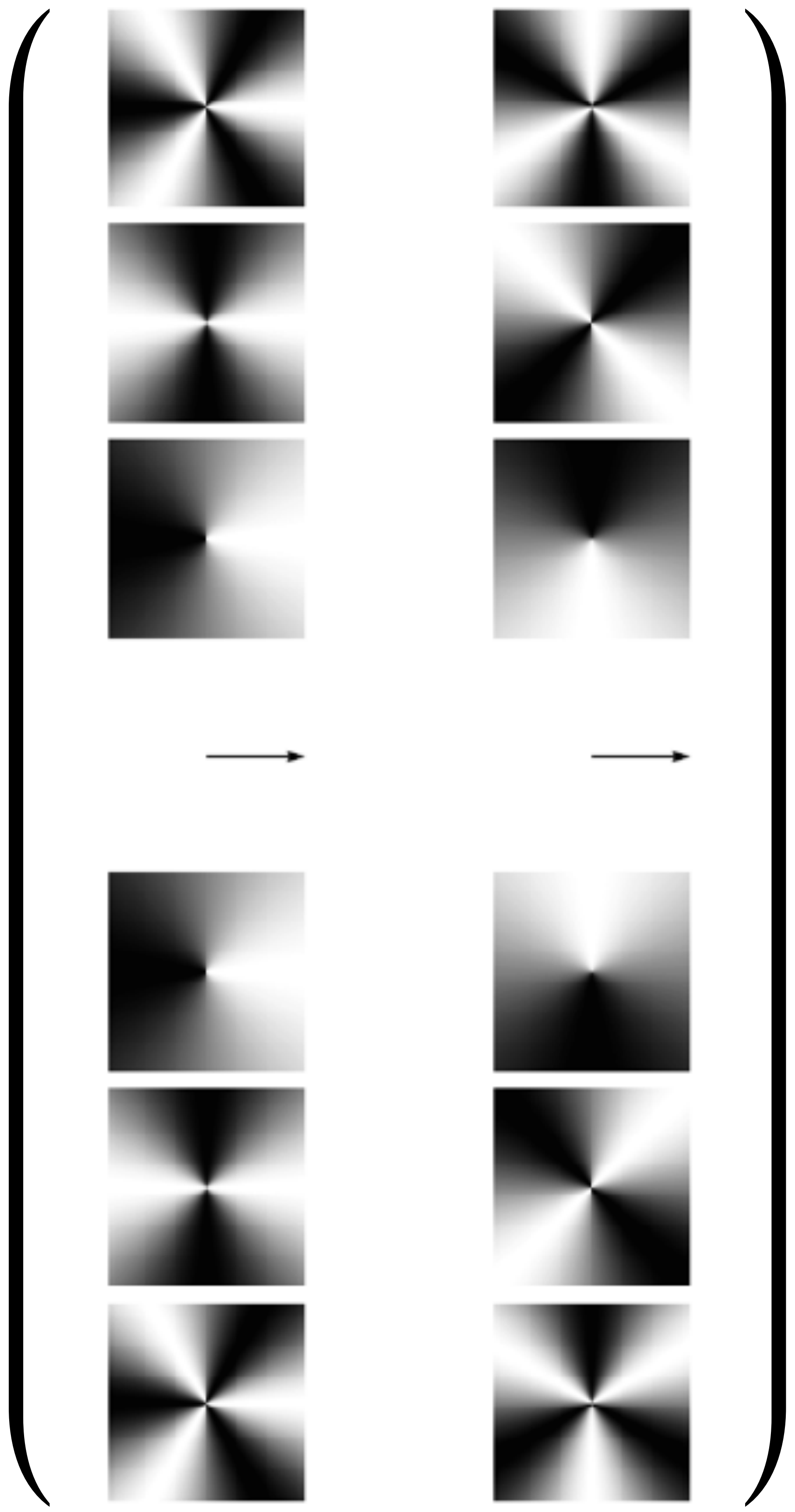


Complex (irreducible) representations

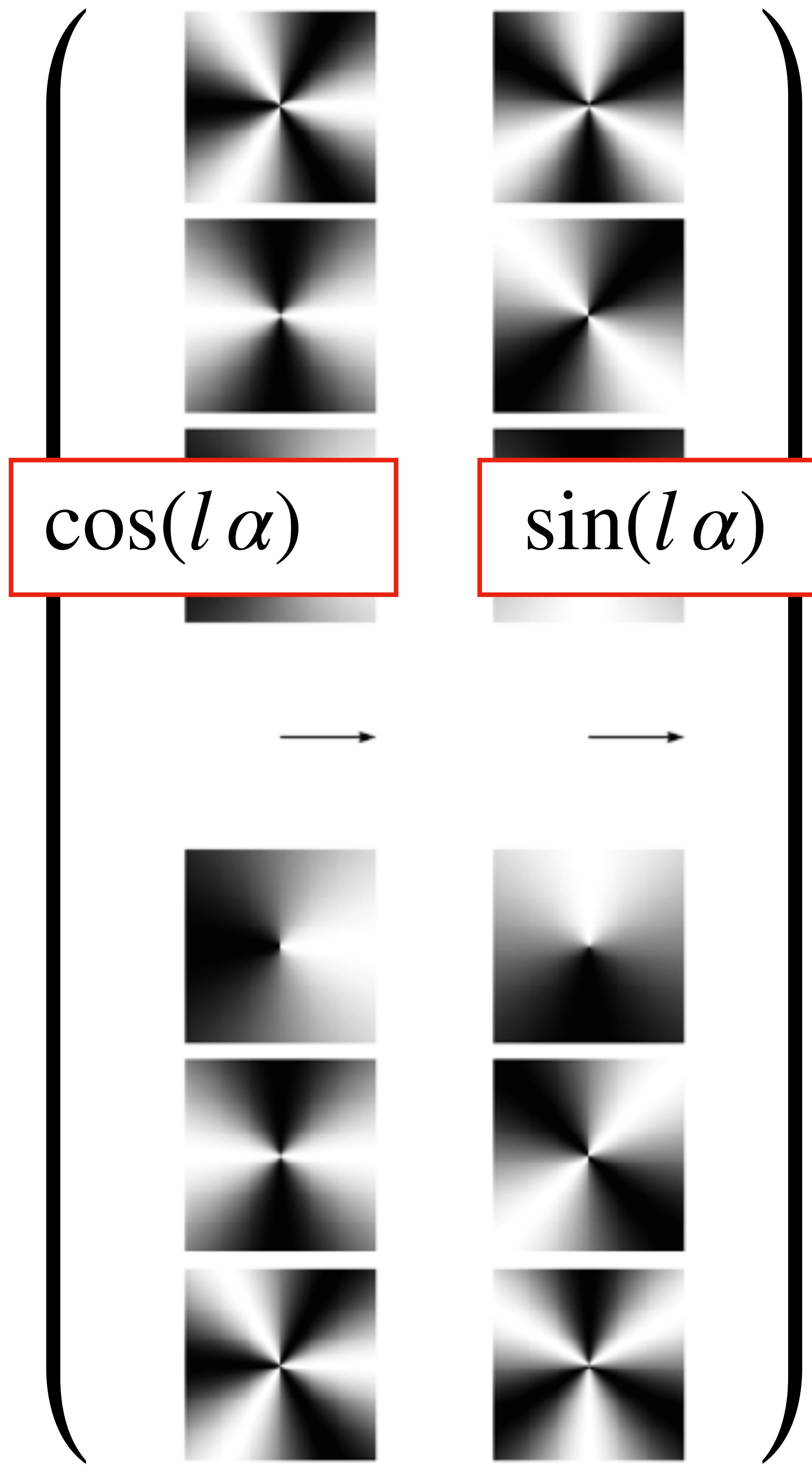
$$Y(\mathbf{R}_\theta^{-1}\mathbf{x}) = \rho(\mathbf{R}_\theta^{-1}) \, Y(\mathbf{x})$$

Re Im



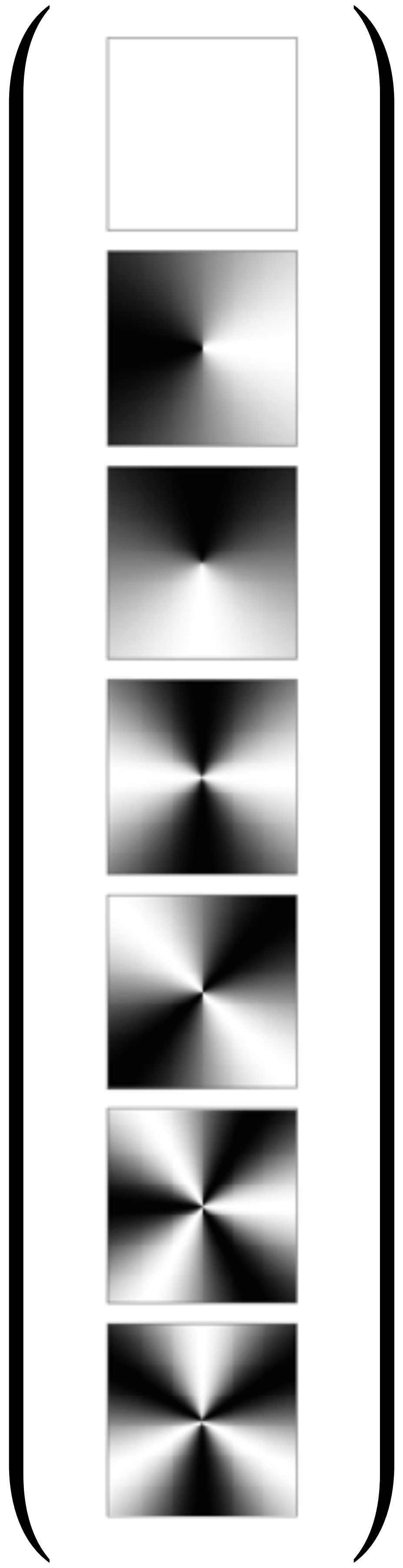
$$=\begin{pmatrix} e^{3i\theta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{1i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-1i\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2i\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-3i\theta} \end{pmatrix}$$

Re Im

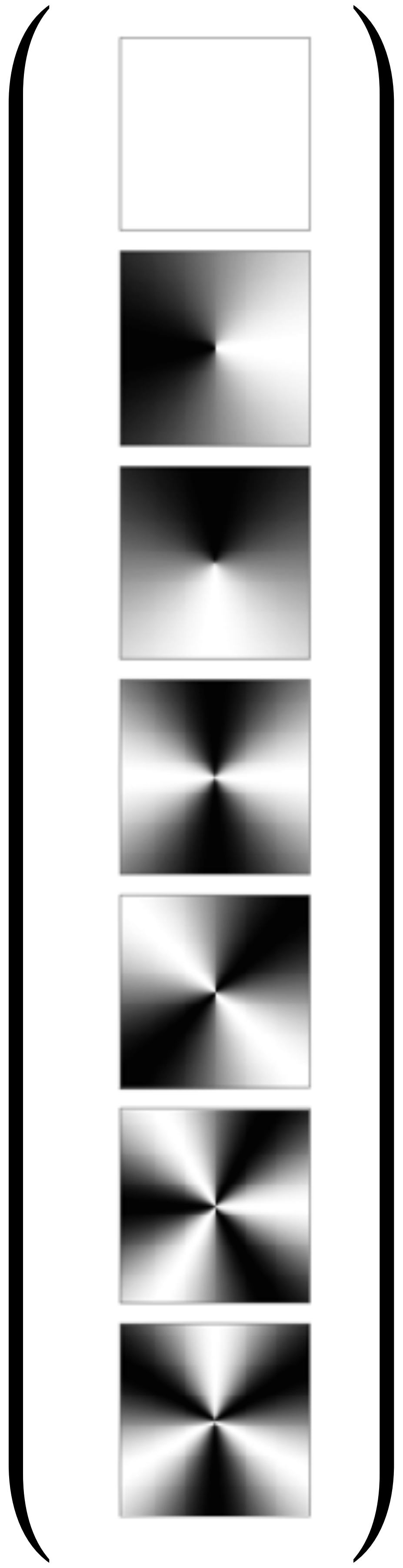


Real (irreducible) representations

$$Y(\mathbf{R}_\theta^{-1}\,\mathbf{x}) \quad = \quad \rho(\mathbf{R}_\theta^{-1}) \quad Y(\mathbf{x})$$



$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos 2\theta & \sin 2\theta & 0 & 0 \\ 0 & 0 & 0 & -\sin 2\theta & \cos 2\theta & 0 & 0 \\ & 0 & 0 & 0 & 0 & \cos 3\theta & \sin 3\theta \\ & 0 & 0 & 0 & 0 & -\sin 3\theta & \cos 3\theta \end{pmatrix}$$



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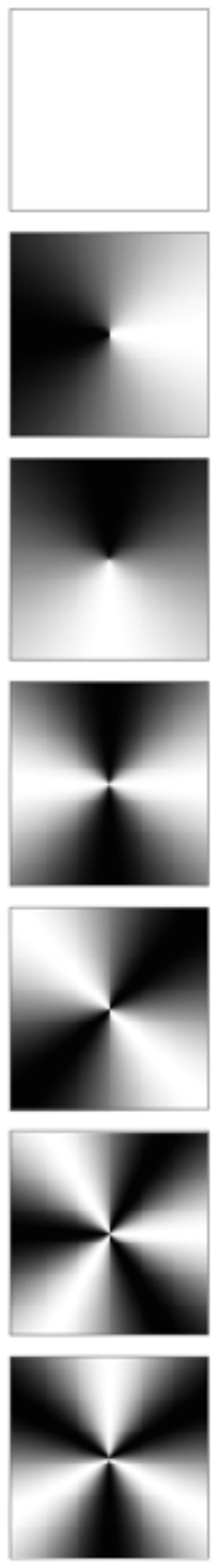

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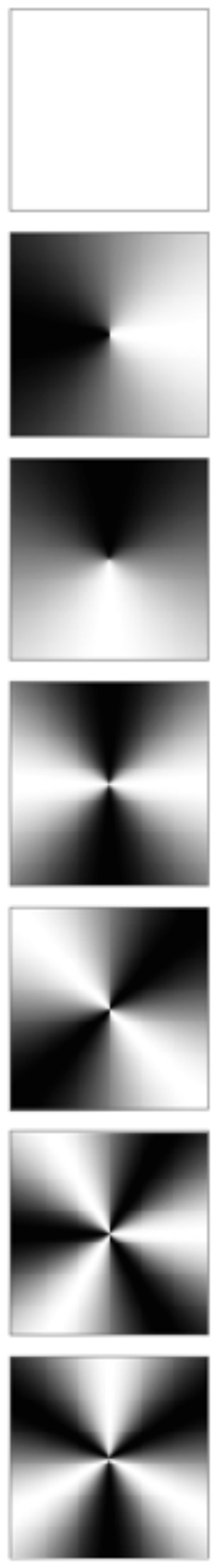
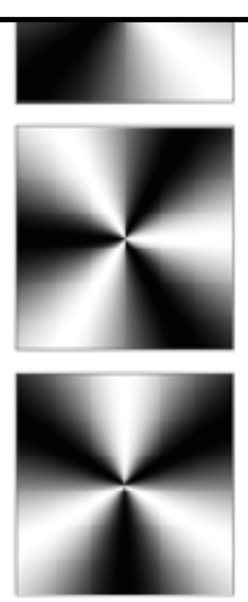


Real (irreducible) representations

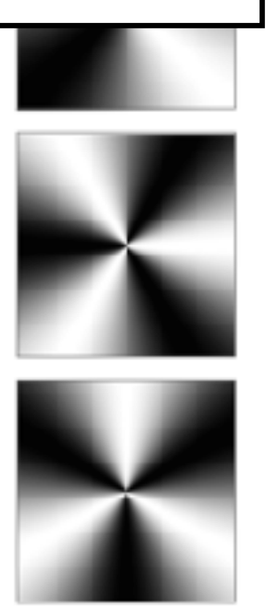
The real basis functions $Y_l(\mathbf{x}) = \begin{pmatrix} \cos(l\alpha) \\ \sin(l\alpha) \end{pmatrix}$ are steerable using $\rho_l(\mathbf{R}_\theta) = \begin{pmatrix} \cos l\theta & -\sin l\theta \\ \sin l\theta & \cos l\theta \end{pmatrix}$

Proof:

$$
\begin{aligned}
Y_l(\mathbf{R}_\theta^{-1}\mathbf{x}) &= \begin{pmatrix} \cos(l(\alpha-\theta)) \\ \sin(l(\alpha-\theta)) \end{pmatrix} \\
&= \begin{pmatrix} \cos(l\alpha + -l\theta) \\ \sin(l\alpha + -l\theta) \end{pmatrix} = \begin{pmatrix} \cos(l\alpha)\cos(-l\theta) - \sin(l\alpha)\sin(-l\theta) \\ \sin(l\alpha)\cos(-l\theta) + \cos(l\alpha)\sin(-l\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos -l\theta & -\sin -l\theta \\ \sin -l\theta & \cos -l\theta \end{pmatrix} \begin{pmatrix} \cos(l\alpha) \\ \sin(l\alpha) \end{pmatrix} \\
&= \rho_l(\mathbf{R}_\theta^{-1})\, Y_l(\mathbf{x})
\end{aligned}
$$



$$
\begin{pmatrix} \cdots & & & & & \\ 0 & 0 & 0 & -\sin 2\theta & \cos 2\theta & 0 & 0 \\ & 0 & 0 & 0 & 0 & \cos 3\theta & \sin 3\theta \\ & 0 & 0 & 0 & 0 & -\sin 3\theta & \cos 3\theta \end{pmatrix}
$$



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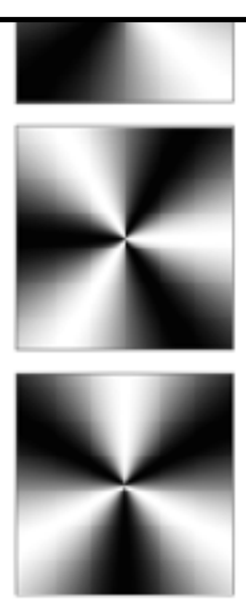
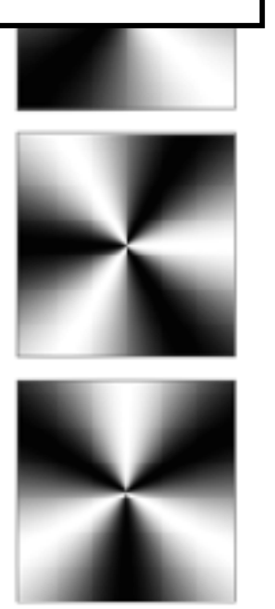
&= \rho_l(\mathbf{R}_\theta^{-1})\, Y_l(\mathbf{x})
\end{aligned}
$$



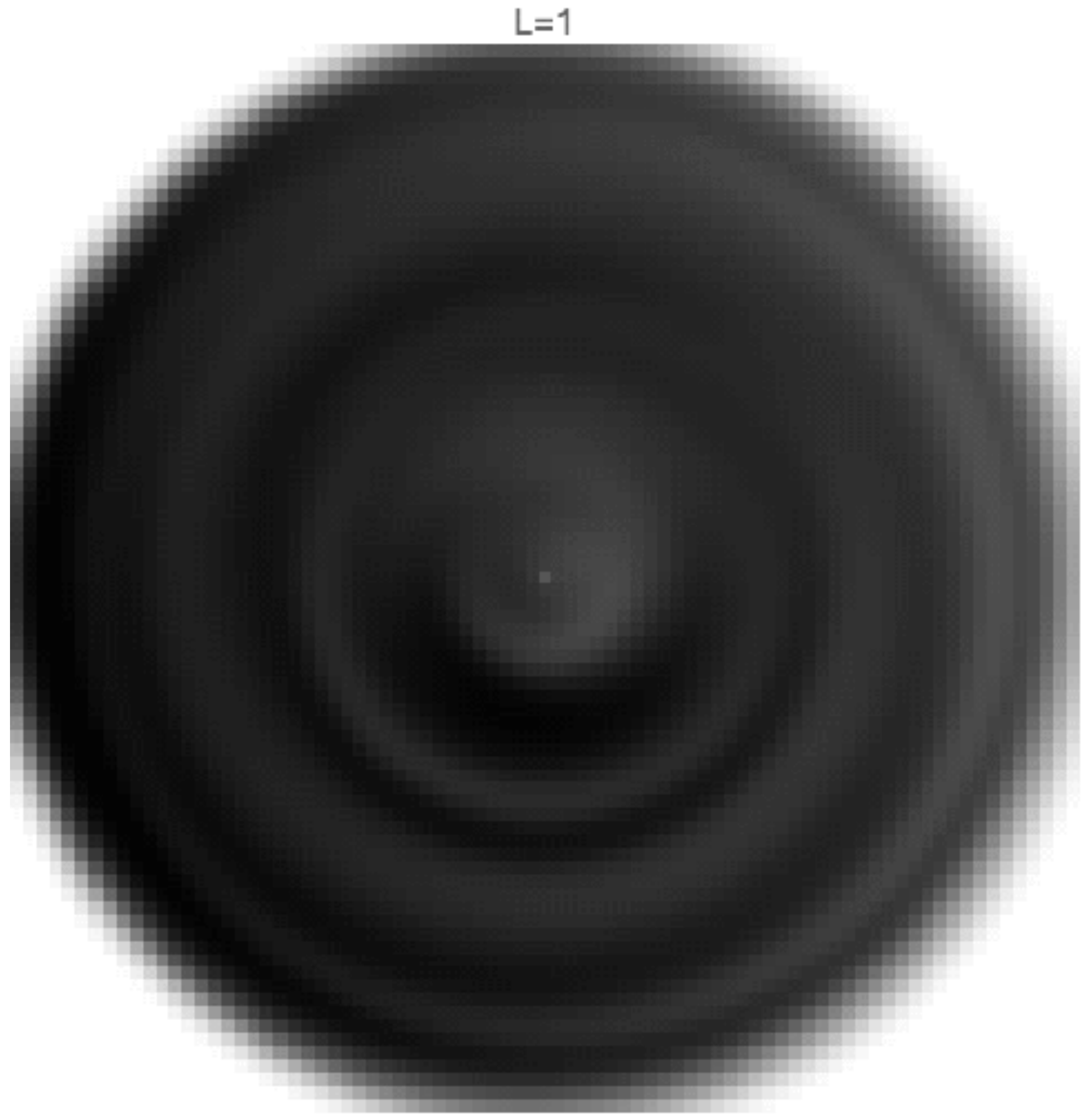
$$
\begin{matrix}
0 & 0 & -\sin 2\theta & \cos 2\theta & 0 & 0 \\
0 & 0 & 0 & 0 & \cos 3\theta & \sin 3\theta \\
0 & 0 & 0 & 0 & -\sin 3\theta & \cos 3\theta
\end{matrix}
$$



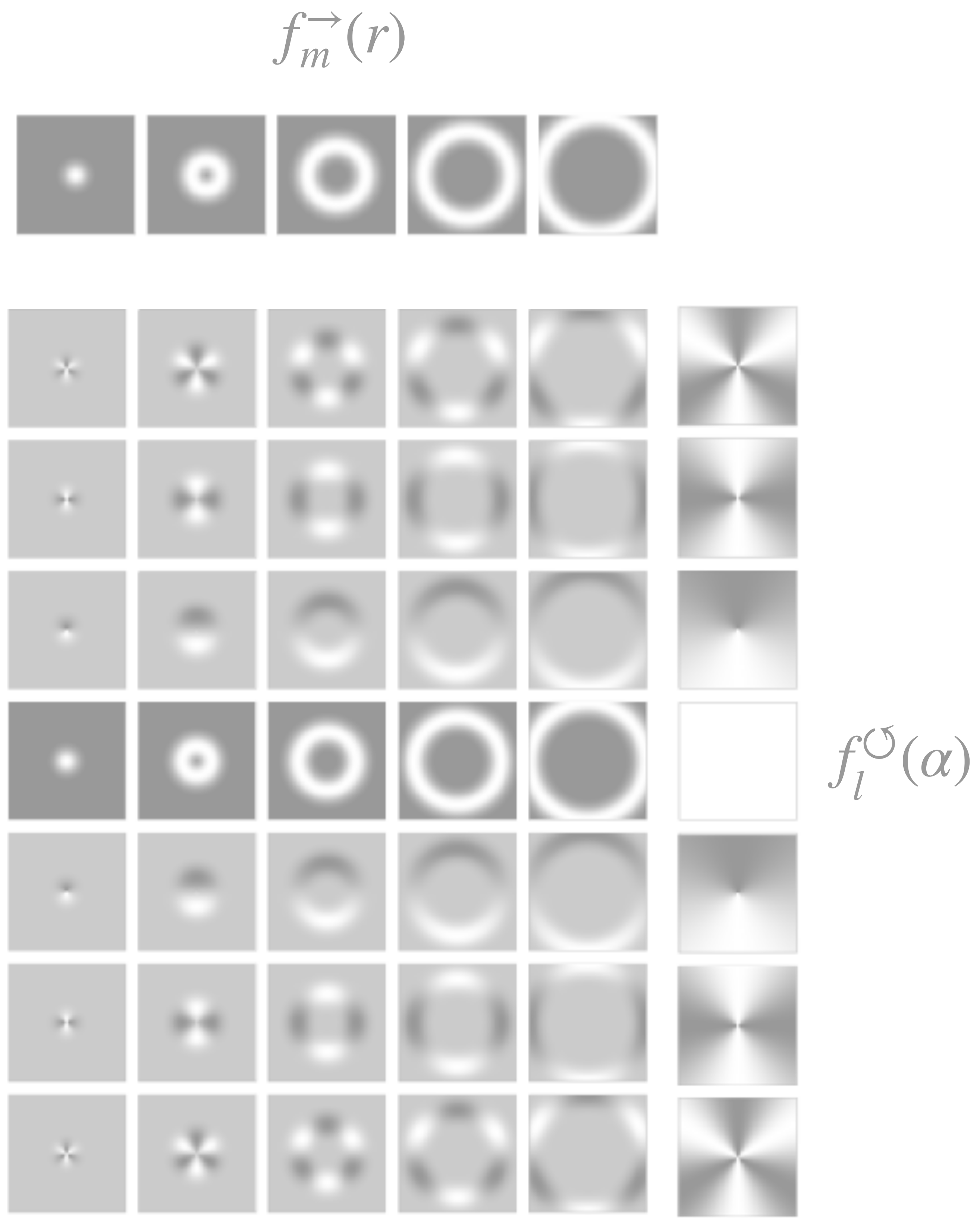
Representing interesting convolution kernels in a steerable basis!

Exercise:

1. Tune the weights $\hat{\mathbf{w}}$ until you get something interesting.
2. Add more detail by increasing maximum frequency!



$k(\mathbf{x} \mid \hat{\mathbf{w}}(r))$



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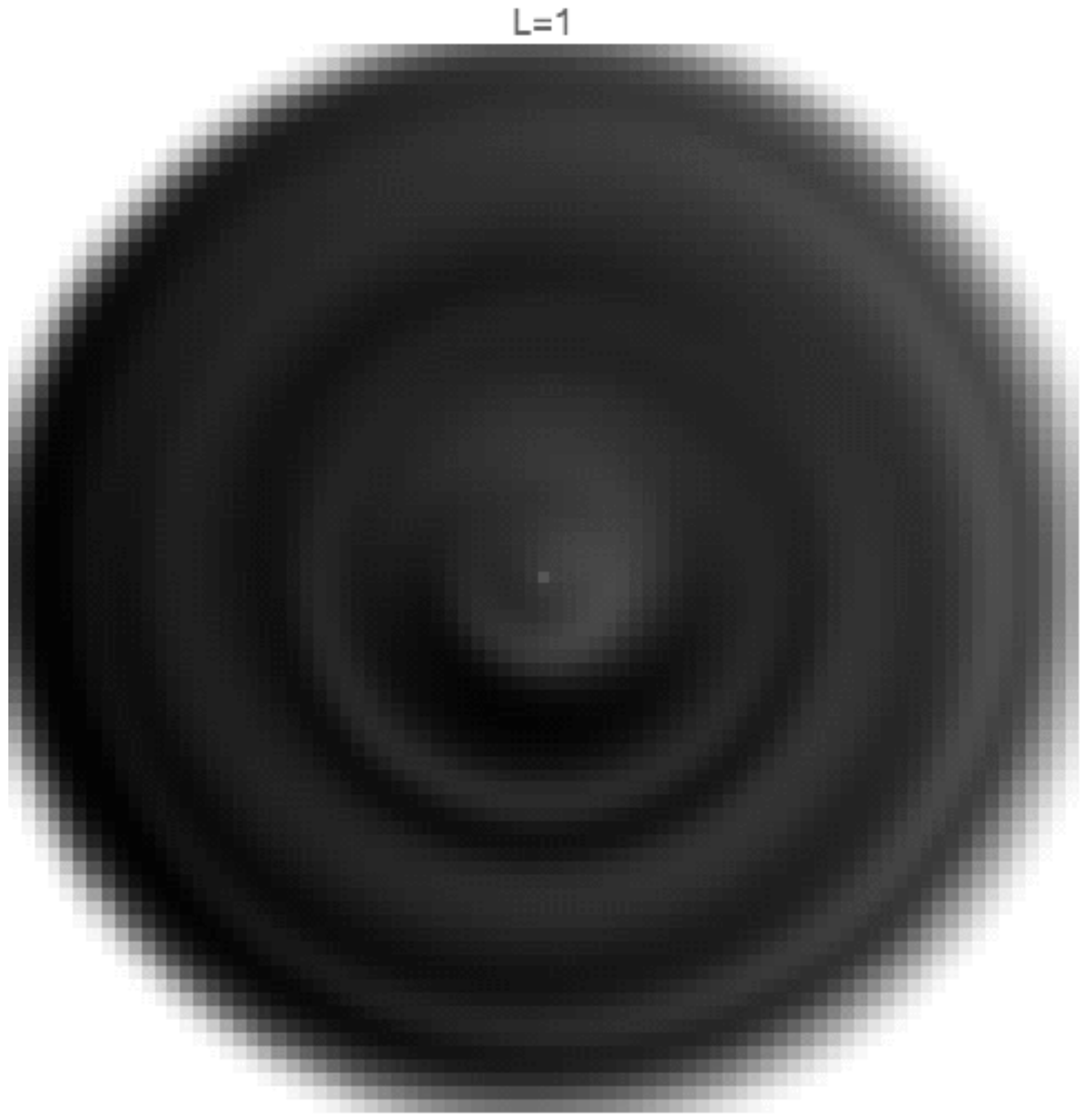
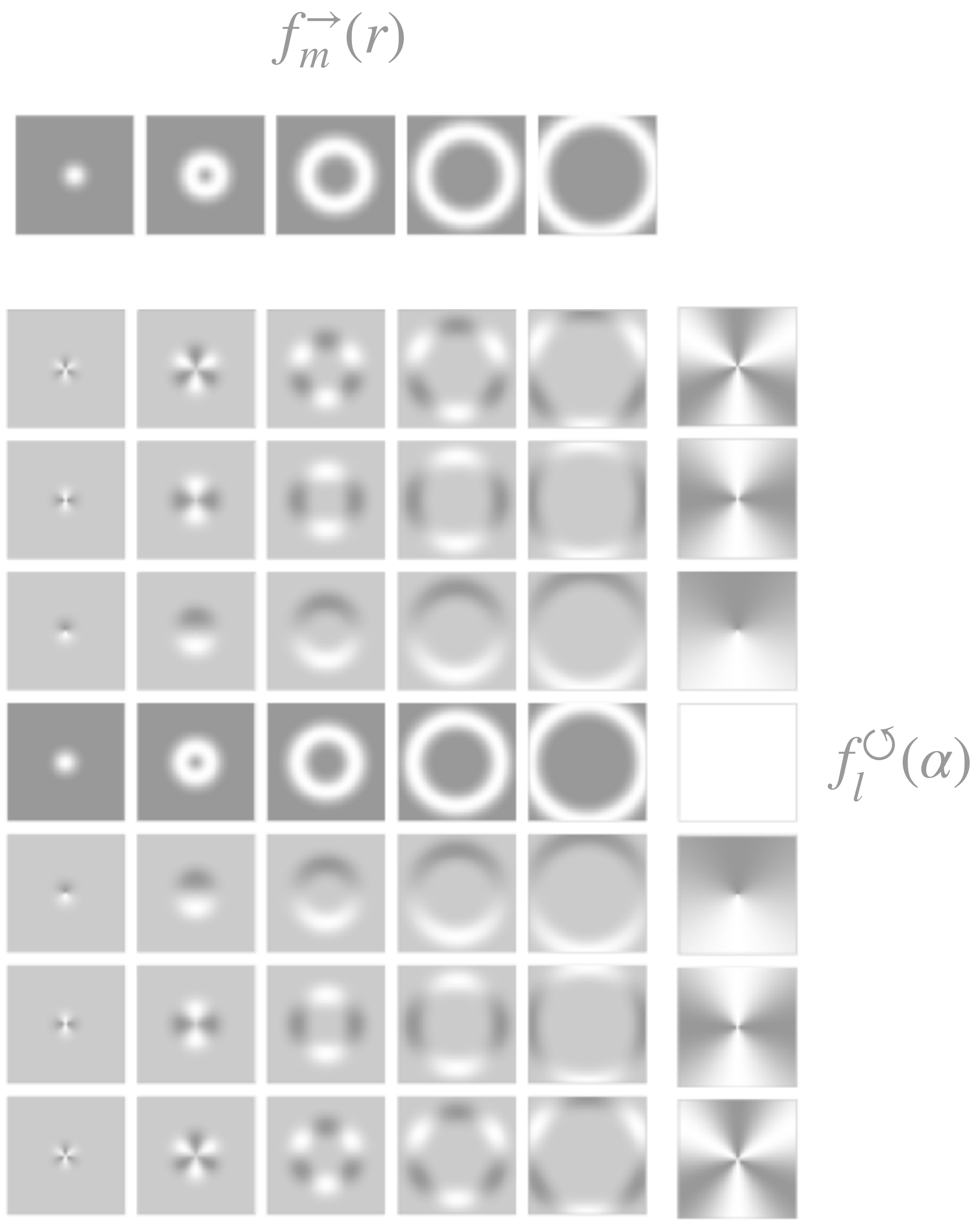


$$k(\mathbf{x} \mid \hat{\mathbf{w}}(r))$$



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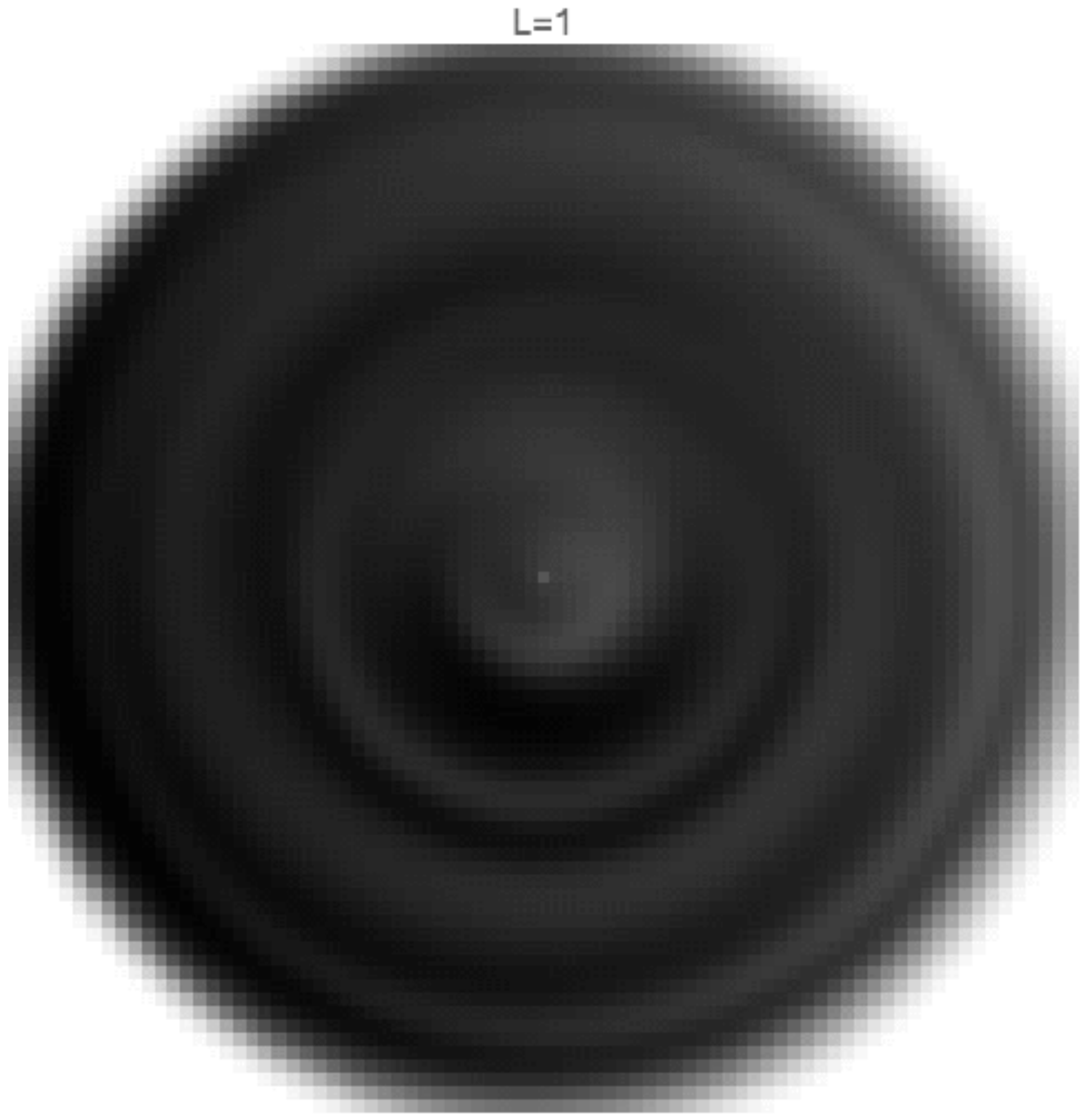

$$k(\mathbf{x} \,|\, \hat{\mathbf{w}}(r))$$

3. Go crazy and steer it by transforming the weights!

$$k(\mathbf{x} \,|\, \rho(\theta)\hat{\mathbf{w}}(r))$$